

## ON THE $p$ TH VARIATION OF A CLASS OF FRACTAL FUNCTIONS

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**ABSTRACT.** The concept of the  $p$ th variation of a continuous function  $f$  along a refining sequence of partitions is the key to a pathwise Itô integration theory with integrator  $f$ . Here, we analyze the  $p$ th variation of a class of fractal functions, containing both the Takagi–van der Waerden and Weierstraß functions. We use a probabilistic argument to show that these functions have linear  $p$ th variation for a parameter  $p \geq 1$ , which can be interpreted as the reciprocal Hurst parameter of the function. It is shown, moreover, that if functions are constructed from (a skewed version of) the tent map, then the slope of the  $p$ th variation can be computed from the  $p$ th moment of a (non-symmetric) infinite Bernoulli convolution. Finally, we provide a recursive formula of these moments and use it to discuss the existence and non-existence of a signed version of the  $p$ th variation, which occurs in pathwise Itô calculus when  $p \geq 3$  is an odd integer.

### 1. INTRODUCTION

Many random phenomena require a description by trajectories that are rougher (or smoother) than the sample paths of continuous semimartingales. A showcase example is the recent observation by Gatheral et al. [10] that the realized volatility of stocks and stock price indices is typically “rough”. To measure the degree of roughness of a function  $f : [0, 1] \rightarrow \mathbb{R}$ , Gatheral et al. [10] and others study expressions of the form

$$(1.1) \quad \sum |f(t_{i+1}) - f(t_i)|^p,$$

where the time points  $t_i$  form a partition of  $[0, 1]$  and  $p \geq 1$  is a parameter. The intuition is that, if the mesh of the partition tends to zero, then there exists a number  $q \in [1, \infty]$  such that the sums in (1.1) will diverge for  $p < q$  and converge to zero for  $p > q$ . This number  $q$  can be regarded as the reciprocal of the Hurst parameter of  $f$ . Clearly, for a typical continuous semimartingale, the corresponding parameter  $q$  would be equal to 2, whereas values larger than 6, or even larger than 10, are observed in [10] for realized volatility trajectories. This observation has spawned a large amount of work on stochastic models for rough volatility. Typically, such models rely on fractional Brownian motion or fractional Ornstein–Uhlenbeck

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processes in describing the rough volatility processes and on rough paths integration theory for the mathematical analysis.

Another recent development concerns a strictly pathwise Itô integration theory for “rough” integrators. For the case of quadratic variation (i.e.,  $q = 2$ , but simultaneously for  $q < 2$ ), this theory goes back to Föllmer [7]. For  $q > 2$ , such a theory was recently developed by Cont and Perkowski [4]; see also Gradinaru et al. [11] and Errami and Russo [5] for related earlier work. It is technically easier than standard rough paths calculus and can still be used to provide a path-by-path analysis of many situations in which stochastic calculus is normally used; see, e.g., [8, 18] and the references therein for several case studies for the case of quadratic variation ( $q = 2$ ). The use of pathwise Itô integration also has the advantage that it is not dependent on probabilistic model assumptions and thus is inherently robust with respect to model risk. This latter point is perhaps particularly important for situations with  $q > 2$ , because there much fewer models are available than in the quadratic variation case.

In this note, our goal is to establish that all functions in a well-studied class of fractal functions have linear  $p$ th variation on  $[0, 1]$ , and thus to establish these functions as valid integrators in the pathwise Itô calculus developed in [4]. This class of fractal functions contains in particular the classical Takagi–van der Waerden and Weierstraß functions. In doing so, we continue the work in [14], where linear  $p$ th variation was established for the special class of Takagi–Landsberg functions. One of our ultimate goals is to provide a class of possible models for “rough” trajectories that allow for an application of Itô calculus and that are not bound by the restrictive assumption of Gaussianity.

Our first main result, Theorem 2.1, shows first that a fractal function  $f$  is either of bounded variation or has non-trivial linear  $q$ th variation, where  $q$  is computed from the parameterization of  $f$ . Moreover, the slope of the  $q$ th variation is identified as the  $q$ th absolute moment of a certain random variable  $Z$ , which in case of the Takagi–van der Waerden functions with even  $b$  has the law of an infinite Bernoulli convolution. Then we turn to a skewed version  $\phi$  of the tent map and investigate a signed version of the  $q$ th variation, which would arise in pathwise Itô calculus if  $q \geq 3$  is an odd integer. In Theorem 3.4 we show that this signed  $q$ th variation may or may not exist, but that it will never vanish as long as  $\phi$  is genuinely skewed. The proof of Theorem 3.4 is based on some auxiliary results on the moments of a general non-symmetric infinite Bernoulli convolution, which may be of independent interest.

In Section 2, we present and prove our general results. In Section 3, we discuss several explicit examples, which include the classical Weierstraß function and the Takagi–van der Waerden functions. Then we discuss the existence and non-existence of the signed  $p$ th variation for a class of functions based on a skewed version of the tent map.

## 2. GENERAL RESULTS

We consider a base function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  that is periodic with period 1, Lipschitz continuous, and vanishes on  $\mathbb{Z}$ . Our aim is to study the function

$$(2.1) \quad f(t) := \sum_{m=0}^{\infty} \alpha^m \phi(b^m t), \quad t \in [0, 1],$$

where  $b \in \{2, 3, \dots\}$  and  $\alpha \in (-1, 1)$ . We exclude the trivial case  $\alpha = 0$ . In this case, the series on the right-hand side converges absolutely and uniformly in  $t \in [0, 1]$ , so that  $f$  is indeed a well defined continuous function. If  $\phi(t) = \sin(2\pi t)$ , then  $f$  is a Weierstraß function. For the tent map,  $\phi(t) = \min_{z \in \mathbb{Z}} |t - z|$ , the function  $f$  belongs to the class of Takagi-van der Waerden functions. For instance, the classical Takagi function [19] has the parameters  $b = 2$  and  $\alpha = 1/2$ . Also the case of a general base function  $\phi$  is well studied; see, e.g., the survey [2] and the references therein.

Here, we analyze the  $p$ th variation of the function  $f$  along the sequence

$$(2.2) \quad \mathbb{T}_n := \{kb^{-n} : k = 0, \dots, b^n\}, \quad n \in \mathbb{N},$$

of  $b$ -adic partitions of  $[0, 1]$  for  $p \geq 1$ . Recall that a function  $f \in C[0, 1]$  admits the continuous  $p$ th variation  $\langle f \rangle_t^{(p)}$  along the sequence  $(\mathbb{T}_n)$ , if for each  $t \in [0, 1]$ ,

$$(2.3) \quad \langle f \rangle_t^{(p)} := \lim_{n \uparrow \infty} \sum_{k=0}^{\lfloor tb^n \rfloor} |f((k+1)b^{-n}) - f(kb^{-n})|^p$$

exists<sup>1</sup> and the function  $t \mapsto \langle f \rangle_t^{(p)}$  is continuous (see, e.g., Lemma 1.3 in [4]). According to Föllmer [7] in the case  $p = 2$  (and simultaneously for any  $p < 2$ ) and Cont and Perkowski [4] in the case of general even  $p$  (and hence for any finite  $p$ ), this notion of  $p$ th variation along a refining sequence of partitions is the key to pathwise Itô integration with integrator  $f$ . Note that for  $p > 1$  the notion of  $p$ th variation is different from the alternative concept of *finite  $p$ -variation* defined in analogy to the total variation by means of a supremum taken over *all* possible partitions of  $[0, 1]$  (see, e.g., [9] or [13]). Also, just as for the usual quadratic variation, the  $p$ th variation of any continuous function, and thus in particular of the sample paths of any continuous stochastic process, depends on the choice of the refining sequence of partitions if  $p > 1$  (see also the discussion in Section 2 of [18]). For the special case  $p = 2$ , certain robustness results are available that allow one to translate results obtained for one refining sequence of partitions to another one that, in a certain sense, is comparable to the first; see [3] and the references therein.

To state our first main result, we fix  $\phi$  and  $b$  and define the coefficients

$$(2.4) \quad \lambda_{m,k} := \frac{\phi((k+1)b^{-m}) - \phi(kb^{-m})}{b^{-m}}, \quad m \in \mathbb{N}, k = 0, \dots, b^m - 1.$$

Next, let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space supporting an independent sequence  $U_1, U_2, \dots$  of random variables with a uniform distribution on  $\{0, 1, \dots, b-1\}$  and define the stochastic processes

$$(2.5) \quad R_m := \sum_{i=1}^m U_i b^{i-1} \quad \text{and} \quad Y_m := \lambda_{m,R_m}, \quad m \in \mathbb{N}.$$

Note that  $R_m$  has a uniform distribution on  $\{0, \dots, b^m - 1\}$ .

**Theorem 2.1.** *Under the assumptions stated above, the following assertions hold:*

- (a) *If  $|\alpha| < 1/b$ , the function  $f$  is of bounded variation.*

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<sup>1</sup>In the sum on the right-hand side of (2.3), the function  $f$  will be formally evaluated at  $1+b^{-n} > 1$  if  $t = 1$  and  $k = b^n$ . To deal with this situation, we will assume here and in what follows that all functions  $f$  defined on  $[0, 1]$  will be extended to  $[0, \infty)$  by putting  $f(t) := f(\min\{t, 1\})$ .

(b) If  $|\alpha| = 1/b$ , then for all  $t \in [0, 1]$ ,

$$(2.6) \quad \lim_{n \uparrow \infty} \sum_{k=0}^{\lfloor tb^n \rfloor} |f((k+1)b^{-n}) - f(kb^{-n})|^p = 0 \quad \text{for all } p > 1.$$

(c) If  $1/b < |\alpha| < 1$ , we let

$$(2.7) \quad Z := \sum_{m=1}^{\infty} (\alpha b)^{-m} Y_m.$$

Then the function  $f$  is of bounded variation on  $[0, 1]$  if and only if  $Z = 0$   $\mathbb{P}$ -a.s. Otherwise, letting  $q := -\log_{|\alpha|} b$ , we have for all  $t \in (0, 1]$ ,

$$(2.8) \quad \lim_{n \uparrow \infty} \sum_{k=0}^{\lfloor tb^n \rfloor} |f((k+1)b^{-n}) - f(kb^{-n})|^p = \begin{cases} 0 & \text{if } p > q, \\ t \cdot \mathbb{E}[|Z|^q] & \text{if } p = q, \\ +\infty & \text{if } p < q. \end{cases}$$

*Remark 2.2.* Let us comment on Theorem 2.1.

- (i) On the one hand, in the case  $|\alpha| = 1/b$  the function  $f$  may be nowhere differentiable and thus not be of bounded variation despite the fact that (2.6) holds. An example is the classical Takagi function, which is obtained by taking  $\phi$  as the tent map,  $b = 2$ , and  $\alpha = 1/2$  (see, e.g., [1]).
- (ii) On the other hand, even if  $1/b < |\alpha| < 1$ , it may happen that the function  $f$  is not a fractal function. Indeed, the following example is given in [2]: let  $\phi(t) = g(t) - \alpha g(bt)$  for some function  $g$  on  $\mathbb{R}$  that is periodic with period 1 and vanishes on  $\mathbb{Z}$ . Then  $f = g$ , and so  $f$  will be smooth if  $g$  is.
- (iii) If  $\phi$  is the tent map,  $b = 2$ , and  $\alpha > 0$ , then  $f$  is often called the Takagi–Landsberg function with Hurst parameter  $H = -\log_b \alpha$ . The corresponding case of Theorem 2.1(c) is contained in Theorem 2.1 of [14];<sup>2</sup> see also Proposition 3.2. Note, moreover, that the  $p$ th variation of fractional Brownian motion with Hurst parameter  $H \in (0, 1)$  vanishes for  $p > 1/H$ , is a non-constant linear function of time for  $p = 1/H$ , and is infinite for  $p < 1/H$  (see, e.g., [15, Section 1.18]). Therefore, (2.8) suggests that also for general  $\phi$ ,  $b$ , and  $\alpha$ , the number

$$H := -\frac{1}{\log_{|\alpha|} b} = -\log_b |\alpha|$$

can be called the *Hurst parameter* of  $f$ . This leads to the following alternative parameterization of the function  $f$  in (2.1) for  $\alpha > 0$ :

$$(2.9) \quad f(t) = \sum_{m=0}^{\infty} b^{-Hm} \phi(b^m t), \quad t \in [0, 1].$$

See Figure 1 for plots of  $\langle f \rangle_1^{(1/H)}$  as a function of  $H$  for two different choices of  $\phi$ .

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<sup>2</sup>Note that in the printed version of [14], there is a factor  $2^{1-1/H}$  missing in the statement of that theorem.

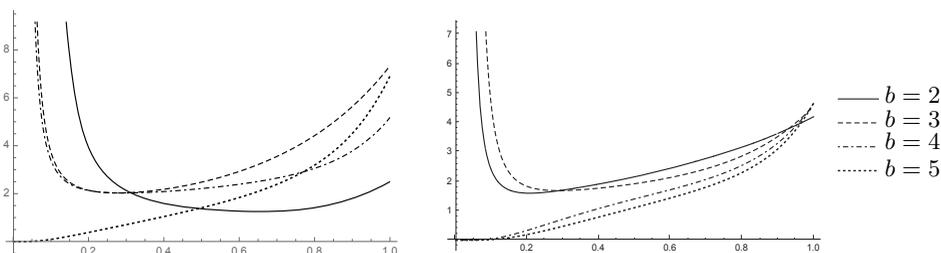


FIGURE 1. The  $(\frac{1}{H})$ th variation  $\langle f \rangle_1^{(1/H)}$  of the function  $f$  in (2.9) as a function of  $H \in (0, 1)$ , for several choices of  $b$ , and for  $\phi(t) = 5 \min_{z \in \mathbb{Z}} |t - z|$  (left) and  $\phi(t) = \frac{1}{2} \sin(2\pi t)$  (right). Note that  $\langle f \rangle_1^{(1/H)}$  tends to 0 or  $+\infty$  as  $H \downarrow 0$ , depending on whether the  $L^\infty$ -norm of  $Z_0 := \sum_{m=1}^\infty b^{-m} Y_m$  is less than or larger than 1.

Now we prepare for the proof of Theorem 2.1, which will be based on two auxiliary lemmas. To this end, we fix  $b \in \{2, 3, \dots\}$  and let  $f$  denote the function defined in (2.1). For  $p \geq 1$ ,  $t \in [0, 1]$ ,  $g \in C[0, 1]$ , and  $n \in \mathbb{N}$ , we define

$$(2.10) \quad V_{p,t,n}(g) := \sum_{k=0}^{\lfloor tb^n \rfloor} |g((k+1)b^{-n}) - g(kb^{-n})|^p.$$

**Lemma 2.3.** For  $n \in \mathbb{N}$ ,  $p \geq 1$  and  $(Y_m)_{m \in \mathbb{N}}$  as in (2.5),

$$(2.11) \quad V_{p,1,n}(f) = (|\alpha|^p b)^n \mathbb{E} \left[ \left| \sum_{m=1}^n (\alpha b)^{-m} Y_m \right|^p \right].$$

*Proof.* Let the  $n$ th truncation of  $f$  be given by  $f_n(t) = \sum_{m=0}^{n-1} \alpha^m \phi(b^m t)$ . It is easy to see that  $f_n(kb^{-n}) = f(kb^{-n})$  for  $k = 0, \dots, b^n$  since  $\phi$  vanishes on  $\mathbb{Z}$ . Hence, for  $k = 0, \dots, b^n - 1$ ,

$$\begin{aligned} f((k+1)b^{-n}) - f(kb^{-n}) &= f_n((k+1)b^{-n}) - f_n(kb^{-n}) \\ &= \sum_{m=0}^{n-1} \alpha^m \left( \phi((k+1)b^{m-n}) - \phi(kb^{m-n}) \right) \\ &= \sum_{m=0}^{n-1} \alpha^m b^{m-n} \lambda_{n-m,k}. \end{aligned}$$

Using the fact that  $R_m$  has a uniform distribution on  $\{0, \dots, b^m - 1\}$ , we hence get

$$\begin{aligned} V_{p,1,n}(f) &= \sum_{k=0}^{b^n-1} \left| \sum_{m=0}^{n-1} \alpha^m b^{m-n} \lambda_{n-m,k} \right|^p \\ (2.12) \quad &= |\alpha|^{np} \sum_{k=0}^{b^n-1} \left| \sum_{m=1}^n (\alpha b)^{-m} \lambda_{m,k} \right|^p = |\alpha|^{np} b^n \mathbb{E} \left[ \left| \sum_{m=1}^n (\alpha b)^{-m} \lambda_{m,R_n} \right|^p \right]. \end{aligned}$$

Next note that for any  $x$  and  $n \geq m$ , due to the periodicity of  $\phi$  and by (2.5),

$$\phi(x + R_n b^{-m}) = \phi\left(x + \sum_{i=1}^n U_i b^{i-1-m}\right) = \phi\left(x + \sum_{i=1}^m U_i b^{i-1-m}\right) = \phi(x + R_m b^{-m}).$$

Hence,  $\lambda_{m,R_n} = \lambda_{m,R_m}$ , and so (2.12) yields (2.11). □

The following simple lemma is a slightly strengthened version of [14, Lemma 3.1].

**Lemma 2.4.** *For  $p \geq 1$  and  $t \in (0, 1]$ , suppose that  $g \in C[0, 1]$  is a function such that  $V_{p,t,n}(g) \rightarrow 0$  as  $n \uparrow \infty$ . Then, for  $h \in C[0, 1]$ , the limit  $\lim_n V_{p,t,n}(h)$  exists if and only if  $\lim_n V_{p,t,n}(g + h)$  exists, and, in this case, both limits are equal.*

*Proof.* Minkowski’s inequality yields that

$$(V_{p,t,n}(h))^{1/p} - (V_{p,t,n}(g))^{1/p} \leq (V_{p,t,n}(g + h))^{1/p} \leq (V_{p,t,n}(g))^{1/p} + (V_{p,t,n}(h))^{1/p}.$$

Passing to the limit  $n \uparrow \infty$  thus yields the “only if” part of the assertion. Rearranging the preceding inequality or replacing  $h$  with  $g + h$  and  $g$  with  $-g$  yields the “if” part.  $\square$

*Proof of Theorem 2.1.* Let  $C$  denote a Lipschitz constant for  $\phi$ . Then, by (2.4),  $|\lambda_{m,k}| \leq C$  and in turn  $|Y_m| \leq C$ .

(a)  $|\alpha| < 1/b$ . Taking  $p = 1$ , Lemma 2.3 yields that

$$0 \leq V_{1,1,n}(f) = \mathbb{E} \left[ \left| \sum_{m=0}^{n-1} \alpha^m b^m Y_{n-m} \right| \right] \leq C \sum_{m=0}^{\infty} (|\alpha|b)^m < \infty.$$

Thus, the sequence  $(V_{1,1,n}(f))$  is bounded uniformly in  $n$ . Next, for  $p = 1$ , the triangle inequality yields that  $V_{1,1,n}(f) \leq V_{1,1,n+1}(f)$  for all  $n$ , and so  $V_{1,1,n}(f)$  converges to a finite limit as  $n \uparrow \infty$ . But since  $f$  is continuous, this limit must coincide with the total variation of  $f$  (see, e.g., Theorem 2 in §5 of Chapter VIII in [16]).

(b)  $|\alpha| = 1/b$ . Here we show that  $V_{p,1,n}(f) \rightarrow 0$  if  $p > 1$ . Indeed, in this case, Lemma 2.3 yields that

$$\begin{aligned} 0 \leq V_{p,1,n}(f)^{1/p} &= (|\alpha|b)^{n/p} \mathbb{E} \left[ \left| \sum_{m=1}^n (\text{sgn } \alpha)^m Y_m \right|^p \right]^{1/p} \\ &\leq (|\alpha|b^{1/p})^n \sum_{m=1}^n \|Y_m\|_{L^p} \leq (|\alpha|b^{1/p})^n nC, \end{aligned}$$

and the rightmost term tends to zero as  $n \uparrow \infty$ . Since  $0 \leq V_{p,t,n}(f) \leq V_{p,1,n}(f)$ , the result holds for all  $t \in [0, 1]$ .

(c)  $|\alpha| > 1/b$ . First we deal with the case  $Z = 0$   $\mathbb{P}$ -a.s. Then

$$\sum_{m=1}^n (\alpha b)^{-m} Y_m = - \sum_{m=n+1}^{\infty} (\alpha b)^{-m} Y_m \quad \mathbb{P}\text{-a.s.}$$

Hence, Lemma 2.3 yields that for  $p = 1$ ,

$$0 \leq V_{p,1,n}(f) \leq C(|\alpha|b)^n \left| \sum_{m=n+1}^{\infty} (\alpha b)^{-m} \right| = \frac{C}{|\alpha b - 1|}.$$

Thus, we conclude as in (a) that  $f$  must be of bounded variation. Once (2.8) will have been established, the converse implication will follow by taking  $p := 1 < q$  in (2.8).

For the remainder of the proof, we suppose that  $\mathbb{P}[Z \neq 0] > 0$ , which implies  $\mathbb{E}[|Z|^p] > 0$  for any  $p \geq 1$ . The fact that  $|Y_m| \leq C$  implies that  $\sum_{m=1}^n (\alpha b)^{-m} Y_m$

converges boundedly to  $Z$  as  $n \uparrow \infty$ . Therefore, for any  $p \geq 1$ ,

$$\mathbb{E} \left[ \left| \sum_{m=1}^n (\alpha b)^{-m} Y_m \right|^p \right] \longrightarrow \mathbb{E}[|Z|^p] < \infty \quad \text{as } n \uparrow \infty.$$

For  $p > q$  we have  $|\alpha|^p b < 1$ , and so Lemma 2.3 yields  $V_{p,1,n}(f) \rightarrow 0$ . This establishes the first case in (2.8) for  $t = 1$ . Since  $0 \leq V_{p,t,n}(f) \leq V_{p,1,n}(f)$ , the result holds for all  $t \in [0, 1]$ .

For  $p = q$  we get  $V_{p,1,n}(f) \rightarrow \mathbb{E}[|Z|^p] > 0$ , and this yields the second case in (2.8) for  $t = 1$ . To establish the assertion also for  $t \in (0, 1)$ , we observe that (2.1) implies that  $f(t) = \phi(t) + \alpha f(bt)$  for  $t \in [0, 1/b]$ . Since  $\phi$  is Lipschitz continuous and  $p > 1$ , one easily gets  $V_{p,t,n}(\phi) \rightarrow 0$  for all  $t$ . Moreover,  $V_{p,1/b,n}(f(b \cdot)) = V_{p,1,n-1}(f) \rightarrow \mathbb{E}[|Z|^p]$ . Hence, Lemma 2.4 yields that

$$\lim_{n \uparrow \infty} V_{p,1/b,n}(f) = |\alpha|^p \lim_{n \uparrow \infty} V_{p,1/b,n}(f(b \cdot)) = |\alpha|^p \mathbb{E}[|Z|^p] = \frac{1}{b} \mathbb{E}[|Z|^p].$$

Iterating this argument gives

$$(2.13) \quad \lim_n V_{p,b^{-\nu},n}(f) = b^{-\nu} \mathbb{E}[|Z|^p] \quad \text{for all } \nu \in \mathbb{N}.$$

Next, for  $\nu \in \mathbb{N}$ ,  $k \in \{0, \dots, b^\nu - 1\}$ , and  $t \in [kb^{-\nu}, 1]$ , the periodicity of  $\phi$  implies that

$$f(t - kb^{-\nu}) = \sum_{m=0}^{\nu-1} \alpha^m (\phi(b^m(t - kb^{-\nu})) - \phi(b^m t)) + f(t) =: g(t) + f(t).$$

Since  $g$  is Lipschitz continuous, it follows from Lemma 2.4 and (2.13) that

$$V_{p,(k+1)b^{-\nu},n}(f) - V_{p,kb^{-\nu},n}(f) \longrightarrow b^{-\nu} \mathbb{E}[|Z|^p] \quad \text{as } n \uparrow \infty.$$

Thus, we get  $V_{p,t,n}(f) \rightarrow t \mathbb{E}[|Z|^p]$  whenever  $t = kb^{-\nu}$  for certain  $\nu \in \mathbb{N}$  and  $k \in \{0, \dots, b^\nu\}$ . A sandwich argument then extends this fact to all  $t \in [0, 1]$ .

Finally, if  $p < q$ , then  $|\alpha|^p b > 1$ , and so  $V_{p,1,n}(f) \rightarrow \infty$  by Lemma 2.3. The analogous fact for  $0 < t < 1$  can be proved as in the final part of the proof of Theorem 2.1 in [14].  $\square$

Combining Theorem 2.1 with [12, Theorem 4.1] and [2, Theorem 2.4] yields immediately the following corollary. See [2, 12] for further conditions that are equivalent to its statements (d)–(f).

**Corollary 2.5.** *In the context of Theorem 2.1(c), the following conditions are equivalent:*

- (a)  $f$  is not of bounded variation.
- (b) For  $q = -\log_{|\alpha|} b$ , the  $q$ th variation of  $f$  is strictly positive.
- (c) The random variable  $Z = \sum_{m=1}^\infty (\alpha b)^{-m} Y_m$  satisfies  $\mathbb{P}[Z \neq 0] > 0$ .

If, in addition,  $\phi$  is piecewise  $C^1$ ,  $\alpha \in (1/b, 1)$ , and  $H := -\log_b \alpha$ , then (a)–(c) are equivalent to each of the following conditions:

- (d)  $f$  is not piecewise  $C^1$ .
- (e)  $f$  is nowhere differentiable.
- (f) The box dimension of the graph of  $f$  is  $2 - H$ .

Now we state a sufficient condition for the properties (a)–(c) in Corollary 2.5. This condition is easy to verify and obviously satisfied for the Takagi–van der Waerden and Weierstraß functions. We refer to Propositions 4.3 and 5.6 in [12] and Theorem 5 in [17] for other sufficient conditions.

**Proposition 2.6.** *Suppose that  $\alpha \in (1/b, 1)$  and that*

$$(2.14) \quad \{0\} \neq \{\phi(b^{-k}) : k \in \mathbb{N}\} \subset [0, \infty).$$

*Then the properties (a)–(c) in Corollary 2.5 are satisfied.*

*Proof.* We show that condition (c) of Corollary 2.5 is satisfied given (2.14). To this end, let  $C$  denote the Lipschitz constant of  $\phi$ . By assumption, there is  $M \in \mathbb{N}$  such that  $\phi(b^{-M}) > 0$ . Choose  $N > M$  and  $\delta > 0$  such that  $C \sum_{m=N}^{\infty} (\alpha b)^{-m} < \phi(b^{-M}) - \delta$ . Then, for  $\omega \in \{U_1 = 0, U_2 = 0, \dots, U_N = 0\}$  and  $m \leq N$ , we have  $Y_m(\omega) = \lambda_{m,0} = b^m \phi(b^{-m}) \geq 0$  and hence

$$\sum_{m=1}^{N-1} (\alpha b)^{-m} Y_m(\omega) \geq (\alpha b)^{-M} Y_M(\omega) \geq \phi(b^{-M}).$$

Therefore,  $|Z(\omega)|$  is bounded from below by

$$\left| \sum_{m=1}^{N-1} (\alpha b)^{-m} Y_m(\omega) \right| - \left| \sum_{m=N}^{\infty} (\alpha b)^{-m} Y_m(\omega) \right| \geq \phi(b^{-M}) - C \sum_{m=N}^{\infty} (\alpha b)^{-m} > \delta.$$

Since  $\mathbb{P}[U_1 = 0, U_2 = 0, \dots, U_N = 0] = b^{-N} > 0$ , we cannot have  $\mathbb{P}[Z = 0] = 1$ .  $\square$

*Remark 2.7.* By considering  $\tilde{\phi}(t) := -\phi(t)$  or  $\hat{\phi}(t) := \phi(-t)$  or  $\bar{\phi}(t) := -\phi(-t)$ , one sees that (2.14) can be replaced by several similar conditions. For instance, requiring (2.14) for  $\bar{\phi}$  is equivalent to the condition  $\{0\} \neq \{\phi(1 - b^{-k}) : k \in \mathbb{N}\} \subset (-\infty, 0]$ . Another easy consequence is that, for any base function  $\phi$  which has a non-vanishing right derivative at 0 or a non-vanishing left derivative at 1, there exists some  $b \in \mathbb{N}$  such that the properties (a)–(c) in Corollary 2.5 are satisfied.

### 3. EXAMPLES AND SIGNED $p$ TH VARIATION

Proposition 2.6 and Theorem 2.1 yield immediately the following corollary.

**Corollary 3.1.** *For a fixed  $\alpha \in (1/b, 1)$  and  $q = -\log_{\alpha} b$ , there exists a constant  $K \in (0, \infty)$  such that the Weierstraß function,*

$$w(t) = \sum_{n=0}^{\infty} \alpha^n \sin(2\pi b^n t), \quad t \in [0, 1],$$

*has linear  $q$ th variation,  $\langle w \rangle_t^{(q)} = tK$ , along the sequence of  $b$ -adic partitions (2.2).*

Now we turn to the class of Takagi–van der Waerden functions, which corresponds to the case in which  $\phi$  is the tent map,

$$(3.1) \quad \phi(t) = \min_{z \in \mathbb{Z}} |t - z|, \quad t \in \mathbb{R}.$$

For  $b = 2$ , the following result is contained in [14, Theorem 2.1].<sup>3</sup> It characterizes the law of  $Z$  in terms of an infinite Bernoulli convolution. Recall that the law of a

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<sup>3</sup>Note that in the printed version of [14], there is a factor  $2^{1-1/H}$  missing in the statement of that theorem.

random variable  $\tilde{Z}$  is a (symmetric) infinite Bernoulli convolution with parameter  $\beta \in (-1, 1)$  if there is an i.i.d. sequence  $(\tilde{Y}_m)_{m=0,1,\dots}$  of  $\{-1, +1\}$ -valued random variables with a symmetric Bernoulli distribution such that  $\tilde{Z} = \sum_{m=0}^{\infty} \beta^m \tilde{Y}_m$ . On the other hand, part (b) of the following proposition yields in particular that if  $b$  is odd, the random variables  $(Y_m)_{m \in \mathbb{N}}$  are no longer independent.

**Proposition 3.2.** *Let  $\phi$  be as in (3.1) and  $1/b < |\alpha| < 1$ .*

- (a) *If  $b$  is even, the random variables  $(Y_m)_{m \in \mathbb{N}}$  form an i.i.d. sequence of symmetric  $\{-1, +1\}$ -valued infinite Bernoulli random variables. In particular, for  $Z$  as in (2.7), the law of  $\tilde{Z} := \alpha b Z$  is the (symmetric) infinite Bernoulli convolution with parameter  $1/(|\alpha|b)$ , and for  $q = -\log_{|\alpha|} b$  we have  $\langle f \rangle_t^{(q)} = t(|\alpha|b)^{-q} \mathbb{E}[|\tilde{Z}|^q]$ .*
- (b) *If  $b$  is odd, the  $(Y_m)_{m \in \mathbb{N}}$  form a Markov chain on  $\{-1, 0, +1\}$  with initial distribution  $[\frac{b-1}{2b}, \frac{1}{b}, \frac{b-1}{2b}]$  and transition matrix*

$$P = \begin{matrix} & -1 & 0 & +1 \\ \begin{matrix} -1 \\ 0 \\ +1 \end{matrix} & \begin{bmatrix} \frac{b+1}{2b} & 0 & \frac{b-1}{2b} \\ \frac{b-1}{2b} & \frac{1}{b} & \frac{b-1}{2b} \\ \frac{b-1}{2b} & 0 & \frac{b+1}{2b} \end{bmatrix} \end{matrix}.$$

*Proof. (a) b is even.* This is a special case of Proposition 3.3.

*(b) b is odd.* The initial distribution of  $Y_1$  is obvious. Next, we observe that  $U_1, \dots, U_m$  can be recovered from  $R_m$  so that

$$(3.2) \quad \sigma(Y_1, \dots, Y_m) \subseteq \sigma(U_1, \dots, U_m) = \sigma(R_m) \quad \text{for } m = 1, 2, \dots$$

Now we consider the event  $\{Y_m = 0\}$  and note that it coincides with  $\{R_m = \frac{b^m - 1}{2}\}$ . But the latter event is equal to  $\{U_1 = \dots = U_m = \frac{b-1}{2}\}$ , which is in turn contained in  $\{R_{m-1} = \frac{b^{m-1} - 1}{2}\} = \{Y_{m-1} = 0\}$ . It follows that

$$\mathbb{P}[Y_m = 0 | R_{m-1}] = \mathbb{P}\left[U_m = \frac{b-1}{2} \Big|_{\{Y_{m-1}=0\}}\right] \stackrel{!}{=} \frac{1}{b} \stackrel{!}{=} \frac{1}{b} \Big|_{\{Y_{m-1}=0\}}.$$

In view of (3.2), this establishes the Markov property for the event  $\{Y_m = 0\}$  and gives the second column of the transition matrix  $P$ .

Next, we have by (2.4) and (2.5) that

$$\{Y_m = 1\} = \left\{ R_m b^{-m} < \left( \frac{b^m - 1}{2} \right) b^{-m} \right\} = \left\{ R_m < \frac{b^m - 1}{2} \right\}.$$

Thus, the independence of  $U_m$  and  $R_{m-1}$  yields that for  $k \in \{0, \dots, b^{m-1} - 1\}$ ,

$$(3.3) \quad \mathbb{P}[Y_m = 1 | R_{m-1} = k] = \mathbb{P}\left[U_m < \frac{b}{2} - \left(k + \frac{1}{2}\right) b^{1-m}\right].$$

If  $0 \leq k < \frac{1}{2}(b^{m-1} - 1)$ , which corresponds to  $Y_{m-1} = 1$ , then

$$(3.4) \quad \frac{b-1}{2} < \frac{b}{2} - \left(k + \frac{1}{2}\right) b^{1-m} \leq \frac{b}{2}.$$

Since there is no integer in the interval  $(\frac{b-1}{2}, \frac{b}{2}]$ , we see from (3.3) that, whenever  $0 \leq k < \frac{1}{2}(b^{m-1} - 1)$ , the probability  $\mathbb{P}[Y_m = 1 | R_{m-1} = k]$  is independent of  $k$  and

equal to  $\mathbb{P}[U_m \leq \frac{1}{2}(b-1)] = (b+1)/(2b)$ . Likewise, for  $\frac{1}{2}(b^{m-1}-1) \leq k \leq b^{m-1}-1$ , which corresponds to  $Y_m = 0$  or  $Y_m = -1$ , we get

$$(3.5) \quad \frac{b}{2} - 1 + \frac{1}{2b^{m-1}} \leq \frac{b}{2} - \left(k + \frac{1}{2}\right)b^{1-m} \leq \frac{b-1}{2}.$$

Again, there is no integer in the interval  $[\frac{b}{2} - 1 + \frac{1}{2b^{m-1}}, \frac{b-1}{2})$ , and so (3.3) implies that

$$\mathbb{P}[Y_m = 1 | R_{m-1} = k] = \mathbb{P}\left[U_m < \frac{b-1}{2}\right] = \frac{b-1}{2b}.$$

Altogether, we have shown that

$$\mathbb{P}[Y_m = 1 | R_{m-1}] = \frac{b+1}{2b} \mathbb{1}_{\{Y_{m-1}=1\}} + \frac{b-1}{2b} \mathbb{1}_{\{Y_{m-1}=0\} \cup \{Y_{m-1}=-1\}}.$$

In view of (3.2), this establishes the Markov property for the event  $\{Y_m = 1\}$  and gives the first column of the transition matrix  $P$ . The analogous result for  $\{Y_m = -1\}$  follows by a symmetry argument.  $\square$

In Remark 1.7 of [4] it is conjectured that, if  $p$  is an odd integer, the following signed  $p$ th variation of  $f$ ,

$$(3.6) \quad \lim_{n \uparrow \infty} \sum_{k=0}^{\lfloor tb^n \rfloor} (f((k+1)b^{-n}) - f(kb^{-n}))^p, \quad t \in [0, 1],$$

will typically vanish for all  $t \in [0, 1]$ . To discuss this conjecture, we will now study the fractal functions  $f$  arising from the following skewed version of the tent map:

$$(3.7) \quad \phi(t) := \begin{cases} t \frac{b}{2\ell} & \text{if } 0 \leq t \leq \ell/b, \\ (1-t) \frac{b}{2(b-\ell)} & \text{if } \ell/b \leq t \leq 1, \end{cases}$$

where  $b \in \{2, 3, \dots\}$  and  $\ell \in \{1, \dots, b-1\}$  are fixed. Then we extend  $\phi$  to all of  $\mathbb{R}$  by periodicity. Note that if  $b$  is even and  $\ell = b/2$ , then  $\phi$  is equal to the standard tent map (3.1), and so the following Proposition 3.3 contains Proposition 3.2(a) as a special case. See Figure 2 for plots of two fractal functions  $f$  corresponding to specific choices of  $b$  and  $\ell$  in (3.7).

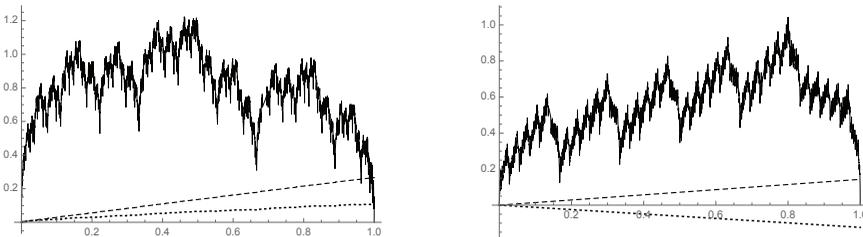


FIGURE 2. Functions  $f$  for the skewed tent map (3.7) for  $b = 3$ ,  $\ell = 1$  (left) and  $b = 6$ ,  $\ell = 5$  (right) with  $\alpha = b^{-1/3}$  in both cases. The dashed and dotted lines, respectively, are the functions  $t \mapsto \sum_{k=0}^{\lfloor tb^6 \rfloor} |f((k+1)b^{-6}) - f(kb^{-6})|^3$  and  $t \mapsto \sum_{k=0}^{\lfloor tb^3 \rfloor} (f((k+1)b^{-6}) - f(kb^{-6}))^3$ .

**Proposition 3.3.** *Let  $\phi$  be as in (3.7) for given  $b \in \{2, 3, \dots\}$  and  $\ell \in \{1, \dots, b-1\}$ . Define  $\mu := -b/(2(b-\ell))$  and  $\nu := b/(2\ell)$ . Then  $(Y_m)_{m \in \mathbb{N}}$  is an i.i.d. sequence of  $\{\mu, \nu\}$ -valued random variables with  $\mathbb{P}[Y_m = \nu] = \ell/b$  and  $\mathbb{E}[Y_m] = 0$ .*

*Proof.* Fix  $m \in \mathbb{N}$  and define the function  $\psi(x) := b^m(\phi(x + b^{-m}) - \phi(x))$ . For  $x \in \{kb^{-m} : k \in \mathbb{Z}\}$ , we have  $\psi(x) \in \{-\frac{b}{2(b-\ell)}, \frac{b}{2\ell}\}$ . More precisely, we have  $\psi(kb^{-m}) = \frac{b}{2\ell} = \nu$  for  $k = 0, \dots, \ell b^{m-1} - 1$  and  $\psi(kb^{-m}) = -\frac{b}{2(b-\ell)} = \mu$  for  $k = \ell b^{m-1}, \dots, b^m - 1$ . Moreover,  $R_m \in \{0, \dots, \ell b^{m-1} - 1\}$  if and only if  $U_m \in \{0, \dots, \ell - 1\}$ . It follows that

$$\mathbb{P}[Y_m = \nu | R_{m-1}] = \mathbb{P}[U_m \in \{0, \dots, \ell - 1\} | R_{m-1}] = \mathbb{P}[U_m \in \{0, \dots, \ell - 1\}] = \frac{\ell}{b}.$$

In view of (3.2), this proves that  $Y_m$  is independent of  $Y_1, \dots, Y_{m-1}$  and has the claimed distribution. □

With the preceding proposition, we are able to prove the following result on the signed  $p$ th variation (3.6) of the functions  $f$  arising from the skewed tent map (3.7).

**Theorem 3.4.** *In the context of Proposition 3.3, suppose that  $\alpha \in (-1, 1)$  is such that  $|\alpha| > 1/b$  and  $q = -\log_{|\alpha|} b$  is an odd integer. Then:*

- (a) *If  $b$  is even and  $\ell = b/2$ , the signed  $q$ th variation (3.6) exists and vanishes identically.*
- (b) *If  $\alpha > 0$ , then the signed  $q$ th variation of  $f$  exists and is given by*

$$(3.8) \quad \lim_{n \uparrow \infty} \sum_{k=0}^{\lfloor tb^n \rfloor} (f((k+1)b^{-n}) - f(kb^{-n}))^q = t \cdot \mathbb{E}[Z^q], \quad t \in [0, 1],$$

where  $Z$  is as in Theorem 2.1. Moreover,  $\mathbb{E}[Z^q]$  is strictly positive if  $\ell < b/2$  and strictly negative if  $\ell > b/2$ .

- (c) *If  $\alpha < 0$  and  $\ell \neq b/2$ , then*

$$(3.9) \quad \lim_{n \uparrow \infty} (-1)^n \sum_{k=0}^{\lfloor tb^n \rfloor} (f((k+1)b^{-n}) - f(kb^{-n}))^q = t \cdot \mathbb{E}[Z^q], \quad t \in [0, 1].$$

*In particular, the signed  $q$ th variation of  $f$  exists only along  $(\mathbb{T}_{2n})_{n \in \mathbb{N}}$  or along  $(\mathbb{T}_{2n+1})_{n \in \mathbb{N}}$ .*

The proof of the preceding theorem is based on the following auxiliary results on the moments of general non-symmetric infinite Bernoulli convolutions. The first is a recursive formula for the moments of a general non-symmetric infinite Bernoulli convolution. A formula for the moments of a standard (symmetric) infinite Bernoulli convolution was given in [6].

**Lemma 3.5.** *Suppose that  $\mu, \nu \in \mathbb{R}$ ,  $p \in (0, 1)$ , and  $(\tilde{Y}_m)_{m \in \mathbb{N}}$  is an i.i.d. sequence of  $\{\mu, \nu\}$ -valued random variables with  $\mathbb{P}[\tilde{Y}_m = \nu] = p$ . For  $\gamma \in (-1, 1)$ , let  $\tilde{Z}$  be the random variable*

$$(3.10) \quad \tilde{Z} = \sum_{m=1}^{\infty} \gamma^m \tilde{Y}_m.$$

Then, for  $k \in \mathbb{N}$ , the  $k$ th moment of  $\tilde{Z}$  is given by the following recursive formula:

$$(3.11) \quad \mathbb{E}[\tilde{Z}^k] = \frac{\gamma^k}{1 - \gamma^k} \sum_{j=0}^{k-1} \binom{k}{j} \left( p\nu^{k-j} + (1-p)\mu^{k-j} \right) \mathbb{E}[\tilde{Z}^j].$$

*Proof.* Note that  $\tilde{Z}$  has the same law as  $\gamma(\tilde{Z} + \tilde{Y}_0)$ , where  $\tilde{Y}_0$  is such that  $(\tilde{Y}_m)_{m=0,1,\dots}$  is an i.i.d. sequence. Conditioning on  $\tilde{Y}_0$  hence yields that for  $k \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{E}[\tilde{Z}^k] &= p\mathbb{E}[(\gamma\tilde{Z} + \gamma\nu)^k] + (1-p)\mathbb{E}[(\gamma\tilde{Z} + \gamma\mu)^k] \\ &= \gamma^k \mathbb{E} \left[ \sum_{j=0}^k \binom{k}{j} p\tilde{Z}^j \nu^{k-j} + \sum_{j=0}^k \binom{k}{j} (1-p)\tilde{Z}^j \mu^{k-j} \right] \\ &= \gamma^k \mathbb{E}[\tilde{Z}^k] + \gamma^k \sum_{j=0}^{k-1} \binom{k}{j} \left( p\nu^{k-j} + (1-p)\mu^{k-j} \right) \mathbb{E}[\tilde{Z}^j]. \end{aligned}$$

This yields (3.11). □

**Example 3.6.** In Theorem 3.4, suppose that  $\alpha = b^{-1/3}$ , so that  $q = 3$ , and then let  $\gamma = 1/(\alpha b)$ ,  $p = \ell/b$ , and  $\mu$  and  $\nu$  as in Proposition 3.3. Our formula (3.11) gives for  $k = 2$  that  $\mathbb{E}[Z^2] = \gamma^2/(1 - \gamma^2)$ . Hence, for  $k = 3$  we get

$$\mathbb{E}[Z^3] = \frac{b^3(b - 2\ell)}{8(b^2 - 1)\ell^2(b - \ell)^2}.$$

When taking  $b = 3$  and  $\ell = 1$  as in the left-hand panel of Figure 2, we get  $\mathbb{E}[Z^3] = 27/256$ . For  $b = 6$  and  $\ell = 5$  as in the right-hand panel of Figure 2, we get  $\mathbb{E}[Z^3] = -875/6912 \approx -0.1266$ .

In the context of Theorem 3.4, the random variables  $Y_m$  are centered. It turns out that in this situation the odd moments of  $Z$  have a common sign as long as  $\alpha > 0$ . This is the content of the following lemma.

**Lemma 3.7.** *In the setting of Lemma 3.5, suppose in addition that  $\mu < 0 < \nu$ ,  $\mathbb{E}[\tilde{Y}_m] = 0$ , and  $\gamma \in (0, 1)$ . Then, for any given odd number  $k \geq 3$ ,*

- (a)  $\mathbb{E}[\tilde{Z}^k] = 0$  if and only if  $\nu = -\mu$ ;
- (b)  $\mathbb{E}[\tilde{Z}^k] > 0$  if and only if  $\nu > -\mu$ ;
- (c)  $\mathbb{E}[\tilde{Z}^k] < 0$  if and only if  $\nu < -\mu$ .

*Proof.* If  $\nu = -\mu$ , then  $\tilde{Z}$  has a symmetric distribution and hence  $\mathbb{E}[\tilde{Z}^k] = 0$ . Thus it suffices to establish the implication “ $\nu > -\mu \Rightarrow \mathbb{E}[\tilde{Z}^k] > 0$ ”, because the corresponding implication in (c) then follows by considering the random variables  $-\tilde{Y}_m$ . So let us assume that  $\nu > -\mu$ . The fact that the  $\tilde{Y}_m$  are centered allows us to assume without loss of generality that  $\nu = 1/p$  and  $-\mu = 1/(1-p)$ , for otherwise we multiply all random variables with  $1/(\nu p)$ . Then (3.11) becomes

$$(3.12) \quad \mathbb{E}[\tilde{Z}^k] = \frac{\gamma^k}{1 - \gamma^k} \sum_{j=0}^{k-1} \binom{k}{j} \left( p^{1+j-k} + (-1)^{k-j}(1-p)^{1+j-k} \right) \mathbb{E}[\tilde{Z}^j].$$

Our assumption  $\nu > -\mu$  implies that  $0 < p < 1/2$  so that  $p^{1+j-k} > (1-p)^{1+j-k}$  for  $j = 0, \dots, k-2$ . Moreover, we have  $\mathbb{E}[\tilde{Z}^j] \geq 0$  for  $j = 0, 1, 2$ . Hence, (3.11) and induction on  $k$  yield that  $\mathbb{E}[\tilde{Z}^k] \geq 0$  for all  $k$ . Since, moreover,  $\mathbb{E}[\tilde{Z}^j] > 0$  for all even  $j$ , the right-hand side of (3.12) is strictly positive for  $k \geq 3$ . □

For the proof of Theorem 3.4, it will be convenient to introduce the following notation. If  $g \in C[0, 1]$ ,  $n \in \mathbb{N}$ , and  $k \in \{0, \dots, b^n - 1\}$ , we write

$$\Delta_{n,k}g := g((k + 1)b^{-n}) - g(kb^{-n}).$$

Next, in analogy to (2.10) we define  $\widehat{V}_{p,n}(g) := \sum_{k=0}^{b^n-1} (\Delta_{n,k}g)^p$  for  $p \in \mathbb{N}$ . Lemma 2.4 does not work for signed  $p$ th variation. We will therefore need the following alternative argument.

**Lemma 3.8.** *Suppose that  $g \in C[0, 1]$  is a function of bounded variation and  $p \in \{2, 3, \dots\}$ . If for some  $q > p$  a function  $h \in C[0, 1]$  has vanishing  $q$ th variation along the  $b$ -adic partitions (2.2), then the limit  $\lim_n \widehat{V}_{p,n}(h)$  exists if and only if  $\lim_n \widehat{V}_{p,n}(g + h)$  exists, and, in this case, both limits are equal.*

*Proof.* Applying Young’s inequality with  $q$  and  $r := q/(q - 1)$  in the third step of the following estimate yields

$$\begin{aligned} & \left| \widehat{V}_{p,n}(g + h) - \widehat{V}_{p,n}(g) \right| \\ & \leq \sum_{k=0}^{b^n-1} \left| (\Delta_{n,k}(g + h))^p - (\Delta_{n,k}g)^p \right| \\ & \leq \sum_{k=0}^{b^n-1} |\Delta_{n,k}g|^p + \sum_{k=0}^{b^n-1} \sum_{\ell=1}^{p-1} \binom{p}{\ell} |\Delta_{n,k}g|^{p-\ell} |\Delta_{n,k}h|^\ell \\ & \leq \sum_{k=0}^{b^n-1} |\Delta_{n,k}g|^p + \sum_{k=0}^{b^n-1} \sum_{\ell=1}^{p-1} \binom{p}{\ell} \left( \frac{1}{r} |\Delta_{n,k}g|^{(p-\ell)r} + \frac{1}{q} |\Delta_{n,k}h|^{\ell q} \right) \\ & = V_{n,1,p}(g) + \sum_{\ell=1}^{p-1} \binom{p}{\ell} \left( \frac{1}{r} V_{n,1,(p-\ell)r}(g) + \frac{1}{q} V_{n,1,\ell q}(h) \right), \end{aligned}$$

where we have used the notation (2.10) in the final step. It is easy to see that our assumptions on  $g$  and  $h$  imply that  $V_{n,1,s}(g) \rightarrow 0$  and  $V_{n,1,\ell q}(h) \rightarrow 0$  as  $n \uparrow \infty$  for all  $s > 1$  and each  $\ell \geq 1$ . This proves the assertion.  $\square$

*Proof of Theorem 3.4.* Our assumption  $|\alpha| > 1/b$  clearly implies that the odd integer  $q$  is larger than or equal to 3. By dropping the absolute values in Lemma 2.3 and its proof, we see that for  $n \in \mathbb{N}$  and  $(Y_m)_{m \in \mathbb{N}}$  as in (2.5),

$$(3.13) \quad \sum_{k=0}^{b^n-1} (f((k + 1)b^{-n}) - f(kb^{-n}))^q = (\operatorname{sgn} \alpha)^n \mathbb{E} \left[ \left( \sum_{m=1}^n (\alpha b)^{-m} Y_m \right)^q \right].$$

Proposition 3.3 and Lemma 3.7 yield that the expectation on the right-hand side vanishes asymptotically if and only if  $b$  is even and  $\ell = b/2$ . In this case, the limit on the left-hand side of (3.13) exists and is zero. In all other cases, the expectation on the right-hand side of (3.13) will converge to  $\mathbb{E}[Z^q] \neq 0$ , and so (3.8) and (3.9) hold for  $t = 1$ . The case of a general  $t \in [0, 1]$  then follows basically as in the proof of Theorem 2.1. One only needs to replace Lemma 2.4 with Lemma 3.8 and to consider the sequences of odd and even  $n$  separately for  $\alpha < 0$ .  $\square$

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