On the First Passage Times of Branching Random Walks in $\mathbb{R}^d$

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Abstract

Given a discrete-time non-lattice supercritical branching random walk in $\mathbb{R}^d$, we investigate its first passage time to a shifted unit ball of a distance x from the origin, conditioned upon survival. We provide precise asymptotics up to $O(1)$ (tightness) for the first passage time as a function of x as $x \to \infty$, thus resolving a conjecture in Blanchet–Cai–Mohanty–Zhang (2024). Our proof builds on the previous analysis of Blanchet–Cai–Mohanty–Zhang (2024) and employs a careful multi-scale analysis on the genealogy of particles within a distance of $\asymp \log x$ near extrema of a one-dimensional branching random walk, where the cluster structure plays a crucial role.

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1 Introduction and main contribution

We examine the first passage times of discrete-time non-lattice branching random walks in $\mathbb{R}^d$. In our setting, a *branching random walk* (BRW) is initiated by a single particle located at the origin $\mathbf{0} \in \mathbb{R}^d$ at time $n = 0$. At each time $n+1$, each particle at time n dies and independently reproduces its descendants according to some probability law on the non-negative integers $\mathbb{N}_0$ with a mean greater than one. Each descendant then independently performs a random walk step, collectively forming the set of particles at time $n+1$. In particular, the underlying genealogy of the particles resembles a supercritical Galton–Watson process. The *first passage time* (FPT) is by definition the first time some particle is present in a prescribed subset of $\mathbb{R}^d$. We refer the readers to a more formal definition of BRW in Section 2.2 of [43]. In this paper, we focus on the first passage times to B_x, the unit ball centered at $(x, 0, \dots, 0) \in \mathbb{R}^d$.

In recent years, numerous studies have focused on BRW in dimension one. Let M_n denote the maximum position of the particles in generation n (or equivalently, at time n). The precise asymptotics of M_n and the limit behavior near frontier have been well-studied by [1, 2, 9, 10, 22, 23, 32], along with the references therein. In particular, under mild conditions, it is shown that there exist constants $c_1, c_2 > 0$ such that $M_n = c_1 n - \frac{3}{2c_2}\log n + O(1)$, where the $O(1)$ term converges in law to a randomly shifted Gumbel distribution. We refer to [3, 40, 43] for notes on BRW and related topics, and Section 2.1 for some selected results useful for our purpose. In terms of FPT, the only works we are aware of are [11, 24], which characterized the law of large numbers and large deviation behavior for the FPT of a one-dimensional BRW with a negative drift.

On the other hand, multi-dimensional BRW have been less studied. However, some of this body of work is reported in [7, 8, 42, 45]. In particular, [7] investigated the asymptotic behavior of the maximum distance from the origin for a spherically symmetric BRW in $\mathbb{R}^d$ and established a precise asymptotic (up to an $O(1)$ factor). In this direction, we also mention recent but earlier studies on the maximum norm of branching Brownian motion (BBM) in $\mathbb{R}^d$ by [6, 26, 27, 33].

Spatial branching processes, including both BRW and BBM, have a wide spectrum of applications ranging from ecology to modeling epidemics ([17, 28, 29, 30], among many others). More recently, motivated by problems from polymer physics, the work [46] initiated the study of the first passage times of spatial branching processes. In particular, it was noted therein that the precise asymptotic for the FPT of BBM follows from established results on multi-dimensional Fisher–KPP equations with boundary value conditions ([16, 19, 39]). The follow-up work [8] further investigated the FPT of BRW and obtained partial results. For the spherically symmetric case, they provide asymptotics up to $O(\log\log x)$ using a particle genealogy approach. For the general case, they achieve asymptotics up to $O(\log x)$ by analyzing random walks in cones. It was conjectured therein that the asymptotic should be precise up to a tight $O(1)$ factor, based on numerics and the following two pieces of theoretical evidence. First, the FPT is exponentially concentrated around its median (Theorem 2 of [8]; see also Lemma 7 below). Second, the analogous asymptotic holds for the BBM (Theorem 1 of [46]).

In this paper, we fully resolve the aforementioned conjecture on the FPT of BRW in $\mathbb{R}^d$ by obtaining the precise asymptotic. We do not require radial symmetry of the process. Our strategy builds in part on the particle genealogy approach in [8] but requires new probabilistic ideas and a finer multi-scale investigation of the genealogy of the one-dimensional BRW. In the subsections below, we state precisely our main result and explain the main ideas underlying our proof.

1.1 Statement of the main results

Consider a discrete-time BRW model with offspring distribution $\{p_i\}_{i\geqslant 0}$, whose mean is denoted by $\rho = \sum_i i\, p_i$ and we assume that $\rho > 1$ (the supercritical case). Recall that for $x \in \mathbb{R}$, B_x denotes the ball of radius one centered at $(x, 0, \dots, 0)$ in $\mathbb{R}^d$. We let V_n denote the collection (i.e. set) of particles at time step n, and $\{\boldsymbol{\eta}_{v,n}(k)\}_{0\leqslant k\leqslant n}$ denote the d-dimensional random walk that leads to $v \in V_n$. Denote by $\eta_{v,n}(k) \in \mathbb{R}$ (resp. $\widehat{\boldsymbol{\eta}}_{v,n}(k) \in \mathbb{R}^{d-1}$) the first coordinate (resp. last $d-1$ coordinates) of $\boldsymbol{\eta}_{v,n}(k)$. We define the FPT τ_x of the BRW to B_x, that is,

$$\tau_x := \min\{n \geqslant 0 : \exists\, v \in V_n,\ \boldsymbol{\eta}_{v,n}(n) \in B_x\}.$$

Let $\boldsymbol{\xi}$ be an $\mathbb{R}^d$-valued random variable representing the increment distribution of the BRW. Denote the first coordinate of $\boldsymbol{\xi}$ by ξ, which is a real-valued random variable. We introduce the large deviation rate function

$$I(x) := \sup_{\lambda>0} \Big(\lambda x - \log \phi_\xi(\lambda)\Big), \tag{1}$$

where $\phi_\xi(\lambda) := \mathbb{E}[e^{\lambda\xi}]$ is the moment generating function for ξ. Consider the following assumptions:[1]

(A1) the offspring distribution has a finite third moment, i.e., $\sum_j j^3 p_j < \infty$;

(A2) the law of $\boldsymbol{\xi}$ is integrable and centered, i.e., $\mathbb{E}[\boldsymbol{\xi}] = \mathbf{0}$;

(A3) the law of $\boldsymbol{\xi}$ is spherically symmetric in $\mathbb{R}^d$,[2] and $\mathbb{P}(\boldsymbol{\xi} = \mathbf{0}) < 1$;

(A4) $\log\rho \in (\mathrm{ran} I)^\circ$, where $(\mathrm{ran} I)^\circ$ is the interior of the range of I. In other words, there exists $c_1 > 0$ such that $I(c_1) = \log\rho$. It can be shown that $c_1 \in (\mathrm{ran}(\log\phi_\xi)')^\circ$. Let $c_2 = I'(c_1)$. (The constants c_1, c_2 arise naturally from the large deviations analysis of the random walk generated by the first-coordinate increment distribution.)

Theorem 1. *Assume (A1)–(A4). Conditioned upon survival, the first passage time for BRW in dimension d to B_x satisfies*

$$\tau_x = \frac{x}{c_1} + \frac{d+2}{2c_1c_2}\log x + O_{\mathbb{P}}(1), \tag{2}$$

where the $O_{\mathbb{P}}(1)$ is tight.

Based on consensus on spatial branching processes, a natural question arises: does the aforementioned $O_{\mathbb{P}}(1)$ term converge in law? If the answer is affirmative, how can we characterize the limit law? Unfortunately, the current techniques developed in this work do not seem sufficient to tackle this question. We leave these challenging questions for future investigation.

Let us now proceed to the non-spherically symmetric case. Let

$$\widehat{I}(\mathbf{x}) := \sup_{\boldsymbol{\lambda}\in\mathbb{R}^d} \Big(\boldsymbol{\lambda}\cdot\mathbf{x} - \log\phi_{\boldsymbol{\xi}}(\boldsymbol{\lambda})\Big) = \sup_{\boldsymbol{\lambda}\in\mathbb{R}^d} \Big(\boldsymbol{\lambda}\cdot\mathbf{x} - \log\mathbb{E}[e^{\boldsymbol{\lambda}\cdot\boldsymbol{\xi}}]\Big) \tag{3}$$

denote the large deviation rate function for $\boldsymbol{\xi}$. We impose the following assumptions:

(A5) the law of $\boldsymbol{\xi}$ is non-lattice in the sense that for all $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $|\mathbb{E}[e^{\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}}]| < 1$;

(A6) $\log\rho \in (\mathrm{ran}\widehat{I}(\cdot,\mathbf{0}))^\circ$, where $\widehat{I}(\cdot,\mathbf{0})$ refers to the function $\widehat{I}$ with the last $d-1$ variables fixed at zero; let $\widehat{c}_1 > 0$ satisfy $\widehat{I}(\widehat{c}_1,\mathbf{0}) = \log\rho$, it holds $(\widehat{c}_1,\mathbf{0}) \in (\mathrm{ran}\nabla\log\phi_{\boldsymbol{\xi}})^\circ$.

Denote by $\mathbf{c}_2 = \nabla\widehat{I}((\widehat{c}_1,\mathbf{0}))$, which is the value of $\boldsymbol{\lambda}$ where the supremum (3) is attained at $\mathbf{x} = (\widehat{c}_1,\mathbf{0})$. The assumption (A5) is crucial for the non-degeneracy of the dimension d of the jumps. More precisely, (A5) implies that any projection of $\boldsymbol{\xi}$ (for instance, $\boldsymbol{\xi}\cdot\mathbf{c}_2$) must also be non-lattice. Both conditions are necessary for the BRW to reach the target ball B_x. Another consequence of (A6) is that $\phi_{\boldsymbol{\xi}}$ is well-defined in a neighborhood of $\mathbf{c}_2$.

Theorem 2. *Assume (A1), (A2), (A5), and (A6). Conditioned upon survival, the first passage time for BRW in dimension d to B_x satisfies*

$$\tau_x = \frac{x}{\widehat{c}_1} + \frac{d+2}{2\,\widehat{c}_1\mathbf{e}_1\cdot\mathbf{c}_2}\log x + O_{\mathbb{P}}(1), \tag{4}$$

where $\mathbf{e}_1 := (1,0,\dots,0) \in \mathbb{R}^d$ and the $O_{\mathbb{P}}(1)$ term is tight.

The assumptions (A5) and (A6) are weaker than (A3) and (A4). Indeed, it is easy to see that (A3) implies (A5), as a consequence of Lemma 24 of [8]; Proposition 21 therein also shows that (A3) and (A4) together imply (A6). Consequently, Theorem 2 is more general than Theorem 1. However, in this paper, we will mainly focus on the proof of Theorem 1. The proof of Theorem 2 is mostly verbatim, where the major changes will be pointed out in Section 4.

[1] These assumptions are almost the same as those in [8], with the only exception of (A1), where we now require a finite third moment on the reproduction law, instead of a finite second moment. This arises from purely technical reasons (see Appendix C.3).

[2] This means that the law of the jump $\boldsymbol{\xi}$ is invariant under any orthonormal transformation in $\mathbb{R}^d$.

Remark 1. Our first passage time is defined in terms of a shifted ball of *radius one*. We expect that the same proof techniques apply to balls of oscillating radii. For instance, one may consider balls of radii $r(x)$ centered at $(x, 0, \dots, 0)$, where $r(x) = O(1)$. The upper and lower bounds for τ_x may be described in terms of $r(x)$.

Remark 2. We also expect that the same technique applies to the branching Brownian motion, hence leading to a probabilistic proof of the asymptotic behavior of the multi-dimensional Fisher–KPP equation with boundary value conditions (as a complement of the works [16, 19]). Remarkably, our probabilistic proof does not specifically rely on the Gaussian structure or any form of symmetry.

1.2 Outline of the proof

In the following, we write $A \ll B$ if there exists a constant $C > 0$ possibly depending on the law of the BRW such that $A \leqslant CB$, and $A \asymp B$ if $A \ll B \ll A$.

Setup and main intuition. Let us assume that the BRW is spherically symmetric, and we condition upon survival. In the spherically symmetric case, let us define

$$t_x := \frac{x}{c_1} + \frac{d+2}{2c_2c_1} \log x, \tag{5}$$

which is the anticipated asymptote of the FPT τ_x (c.f. (2)), where we recall that $I(c_1) = \log \rho$, $I'(c_1) = c_2$, and I is the rate function for the first coordinate of $\boldsymbol{\xi}$. Denote by M_n the maximum at level n of a BRW with jump ξ, and its asymptote $m_n := c_1 n - \frac{3}{2c_2} \log n$.

We first explain the intuition behind the asymptote (5) and the main difficulties behind the proof of Theorem 1. By inverting the expression of m_n, one expects that the FPT to $\mathbb{H}_x := [x, \infty) \times \mathbb{R}^{d-1}$ is around

$$t_x^{(1)} := \frac{x}{c_1} + \frac{3}{2c_2c_1} \log x.$$

Indeed, $m_{t_x^{(1)}} = x + O(1)$. As a result of the local CLT (under a change of measure and possibly under barrier constraint; see Lemmas 9 and 14), each particle that arrives in $\mathbb{H}_x$ has a chance around $x^{-\frac{d-1}{2}}$ landing in B_x. Suppose that we wait until the first $x^{\frac{d-1}{2}}$ particles hitting $\mathbb{H}_x$, then on average there would be $\asymp 1$ particles found in B_x. Meanwhile, it is certainly not the case that the displacements in the last $d-1$ coordinates are almost independent among the ($x^{\frac{d-1}{2}}$ many) *frontier particles* in the first coordinate, i.e., those that travel fast in the first dimension. This is because the frontier particles may not be separated until very late, resulting in a strong dependence between displacements in the last $d-1$ coordinates. In other words, to study bounds of τ_x, one needs to understand the genealogies (or the dependence structure) of the (roughly $(\log x)x^{\frac{d-1}{2}}$ many) frontier particles that are the fastest in the first dimension around time t_x. Here, the quantity $(\log x)x^{\frac{d-1}{2}}$ is the asymptotic of the number of particles in $\mathbb{H}_x$ at time t_x, which has been computed by [8]; see Lemma 6 below.

Prelude: introducing the role of clusters in the genealogy of the frontier particles. The goal of the paragraphs in this subsection is to expose a cluster structure that is useful in our analysis. We do this by introducing a conditional probability (see (6) below) that captures key elements in our analysis, although we do not directly study this probability in our future development, it is useful as a device to quickly see the main ingredients that will come in to play. As we shall discuss later, the (non-trivial) genealogy of the frontier particles is reasonably well understood. Suppose that we condition both, on the genealogy of the BRW up to time t_x and BRW but only its projection onto the first coordinate. Consider the set $T = \{v \in V_{t_x} : \boldsymbol{\eta}_{v,t_x}(t_x) \in \mathbb{H}_x\}$, which is measurable and, by the above discussion, it has cardinality $\#T \asymp (\log x)x^{\frac{d-1}{2}}$ with high probability. Next, define the (conditioned) locations of the particles $v \in T$ in the rest $d-1$ dimensions, written as $\{\mathbf{X}_v\}_{v\in T} := \{\widehat{\boldsymbol{\eta}}_{v,t_x}(t_x)\}_{v\in T}$. Recall that our goal is to find a particle in B_x at time t_x, which is, roughly speaking, equivalent to finding a particle $v \in T$ such that $\|\mathbf{X}_v\| \leqslant 1$.[3] Here and later, we use $\|\cdot\|$ to denote the Euclidean norm. We are then reduced to the following problem (after conditioning on the genealogy and the first BRW coordinate of the frontier particles): given an $\mathbb{R}^{d-1}$-valued stochastic process $\mathbf{X} = \{\mathbf{X}_v\}_{v\in T}$ where T is a finite set, how to characterize the probability

$$\mathbb{P}(\exists\, v \in T,\ \|\mathbf{X}_v\| \leqslant 1) \tag{6}$$

[3] This is because a non-trivial proportion of particles in T will be located in $[x, x + 1/2]$ in the first coordinate, so we may simply look at particles in $\mathbb{H}_x$ at time t_x.

up to multiplicative constants, in terms of the dependence structure of $\{\mathbf{X}_v\}_{v\in T}$ (keep in mind that the probability in (6) is conditional, as stated earlier, so bounds on (6) are to be understood as high-probability bounds). Intuitively, $\mathbf{X}_v$ and $\mathbf{X}_w$ are strongly dependent if v and w are separated very late in the underlying genealogy, and vice versa.

While the problem (6) appears fundamental, we are unaware of a solution even in special cases.[4] There are a few natural ideas for upper and lower bounding the quantity (6):

A) To bound $\mathbb{P}(\exists\, v \in T,\, \|\mathbf{X}_v\| \leqslant 1)$ from below, pick a subset $T' \subseteq T$ such that $\mathbf{X}_v$ and $\mathbf{X}_w$ are approximately independent for all $v, w \in T'$, $v \neq w$, and use $\mathbb{P}(\exists\, v \in T',\, \|\mathbf{X}_v\| \leqslant 1)$ as a lower bound.

B) To bound $\mathbb{P}(\exists\, v \in T,\, \|\mathbf{X}_v\| \leqslant 1)$ from above, partition T into "well-separated blocks" $T_1, \dots, T_m$ that are approximately independent (i.e., for any $v \in T_i$, $w \in T_j$, $i \neq j$, $\mathbf{X}_v$ and $\mathbf{X}_w$ are roughly independent) and such that the "size" of each block is small (and hence for each $1 \leqslant j \leqslant m$, $\mathbb{P}(\exists\, v \in T_j,\, \|\mathbf{X}_v\| \leqslant 1)$ is small).[5]

In practice, these bounds work well if the set T has the following *cluster structure*: T can be partitioned into well-separated blocks/clusters with very small sizes (for instance, consider the extreme case where $\{\mathbf{X}_v\}_{v\in T}$ forms an i.i.d. sequence, or an identical sequence). Obtaining a characterization of (6) for a general process $\{\mathbf{X}_v\}_{v\in T}$ is difficult, but is not necessary for our purpose since we are interested in the special class of processes $\mathbf{X}$ that describes the genealogy of the extremal particles for a one-dimensional BRW. This highlights the importance of understanding the cluster structure of extremal particles.

The cluster structure of extremal particles. Conditioning on the particle genealogy of those in $\mathbb{H}_x$ at time t_x, we effectively obtain a stochastic process $\{\mathbf{X}_v\}_{v\in T}$ describing the displacements in the last $d-1$ dimensions of the frontier particles. In this case, we give upper and lower bounds for (6) that may not match in general, but surprisingly, they coincide (up to multiplicative constants) with high probability. This is because of the following nice feature of the *one-dimensional* BRW:

$$\text{the frontier particles are most likely to be separated either} \atop \textit{very early} \text{ or } \textit{very late} \text{ in the underlying genealogy.} \tag{7}$$

A precise formulation concerning a number of $O(1)$ many frontier particles can be found in Theorem 4.5 of [34] in the context of BRW, and Theorem 2.1 of [4] in the context of BBM. Loosely speaking, as $n \to \infty$, the particles beyond m_n at time n are separated in either the first $O(1)$ steps or the last $O(1)$ steps with high probability. This observation can be generalized to the study of around $(\log x)x^{\frac{d-1}{2}}$ many frontier particles (or particles beyond x at time t_x), using an extension of Proposition 8 of [8]. For $0 \leqslant n \leqslant t_x$, we define the *production number* P_n as the number of particles at level n that allows a descendant beyond x in the first coordinate, at time t_x. For instance, $P_0 = 1$ and $P_{t_x} \asymp (\log x)x^{\frac{d-1}{2}}$ with high probability. We will prove in Proposition 8 that P_n only has non-trivial increase on the intervals $n \in [0, O((\log x)^2)] \cup [t_x - O((\log x)^2), t_x]$; see Figure 1 for an illustration. In other words, the *majority* of the (roughly $(\log x)x^{\frac{d-1}{2}}$ many) particles beyond x at time t_x are either separated in the first $O((\log x)^2)$ steps or the last $O((\log x)^2)$ steps with a non-trivial probability.[6]

The effect (7) is related to *entropic repulsion*, which describes the phenomenon that a typical path leading to maximum lies well below the interpolating line in most intermediate times because those locations well below the interpolating line are not favorable as a branching location that leads to another extremal particle. Thus, the overwhelming majority of the leading frontier particles will exhibit the entropic repulsion phenomenon. This means that early on in the history of the BRW, leading particles' induced "clusters" in the genealogy start being formed early on (within $(\log x)^2$ time), and by time t_x there are roughly $x^{(d-1)/2}$ many clusters[7] that are well separated in the metric of the tree generated by the genealogy. More comprehensive discussions of the leading particles' genealogy can be found in [4, 12, 13, 21]. However, a major difference is that these works focused on $O(1)$ many frontier particles of the BBM, instead of $\asymp (\log x)x^{\frac{d-1}{2}}$ many frontier particles of the BRW. Another technical difference is that their clusters classify *all* particles by the genealogical distance and are re-centered by the maximum location in each cluster; in our case, we focus only on particles beyond a certain threshold (instead of collecting all of them).

[4] A particularly interesting problem would be, for instance, assuming $\mathbf{X}$ is one-dimensional centered Gaussian (and hence written as X), characterize (6) in terms of the distance $d_X(v,w) = \sqrt{\mathbb{E}[(X_v - X_w)^2]}$.

[5] This can be viewed as a simplified version of the generic chaining technique [41].

[6] Due to the nature of the second moment method we apply, we cannot conclude a with-high-probability statement. While we resolve this issue by using the exponential concentration of the FPT (Lemma 7 below), we conjecture that such a property holds with high probability.

[7] Note that this is a rather coarse approximation, each leading particle could form early on itself a random number of clusters, but the expectation of this number is finite.

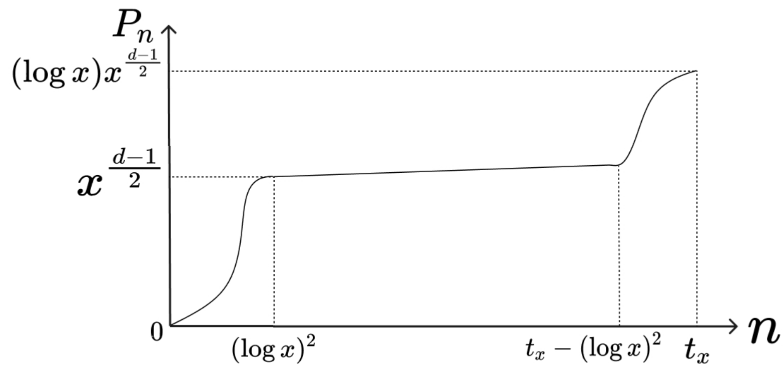

Figure 1: Typical growth pattern of the production number P_n. The plot in the interval $[(\log x)^2, t_x-(\log x)^2]$ is almost flat, reflecting the fact that the only non-trivial increase arises from $n \ll (\log x)^2$ or $t_x - n \ll (\log x)^2$.

Turning to the picture of the process $\{\mathbf{X}_v\}_{v\in T}$, this suggests that the set T enjoys the cluster structure suggested earlier. Figure 2 illustrates this phenomenon. Roughly speaking, each element in the partition of T into blocks then corresponds to a collection of particles that do not separate until time $t_x - O((\log x)^2)$, or equivalently until $O((\log x)^2)$, meaning that these blocks are well-separated. As a consequence of Proposition 8, the number of such blocks is around $x^{\frac{d-1}{2}}$. By a conditional local CLT we establish below (Lemma 9), each block has a chance of around $x^{-\frac{d-1}{2}}$ of having a particle located in B_x, and these events for each block are approximately independent (a key ingredient in A)). The upper bound for τ_x then follows, which we elaborate on in Section 2.3.

On the other hand, turning to the block-size mentioned in B), it is nontrivial to show that the sizes of these individual clusters are small. One can show that each cluster has size $\ll \log x$, most have size $\ll 1$, and on average has cardinality $\asymp \log x$, but these pieces of information are not sufficient to conclude a matching upper bound for (6). To proceed further, one needs the following crucial observation. There are two fundamentally different ways to upper bound the "sizes" of individual clusters:

- The size of a cluster T_j is small if it contains very few elements (i.e., its cardinality is small). In this case, we use the union bound to obtain

$$\mathbb{P}(\exists\, v\in T_j,\ \|\mathbf{X}_v\|\leqslant 1) \ll x^{-\frac{d-1}{2}}\#T_j. \tag{8}$$

- The size of a cluster T_j is small if its "dispersion" is small, precisely, if $\mathbb{E}[\sup_{v,w\in T_j}\|\mathbf{X}_v-\mathbf{X}_w\|]$ is small. In this case, we expect that

$$\mathbb{P}(\exists\, v\in T_j,\ \|\mathbf{X}_v\|\leqslant 1) \ll x^{-\frac{d-1}{2}}\mathbb{E}\Big[\sup_{v,w\in T_j}\|\mathbf{X}_v-\mathbf{X}_w\|\Big]. \tag{9}$$

In summary, our goal is to show that, with high probability, the random set T exhibits the cluster structure explained, and most of the $\asymp x^{\frac{d-1}{2}}$ many clusters satisfy the following: either its cardinality is small, or its dispersion is small. Equivalently, consider the collection $\mathcal{P}$ of particles at time $\widetilde{t}_x := t_x - (\log x)^2$ that lead to a descendant beyond x at time t_x (in the first coordinate). We need to show that with high probability, for *most* particles in $\mathcal{P}$, either each of these has very few descendants reaching x at time t_x, or all of its descendants that reach x have a very young common ancestor (so the dispersion is controlled with the help of a suitable conditional local CLT). Achieving this goal is the most technical part of this paper. Below we attempt to sketch the intuition without going into too many details.

Bounding the size of the clusters. The plan is to condition on an ancestor at time $\widetilde{t}_x$ (as well as the first coordinate of its location), discretize the space, and perform the following multi-step conditioning analysis of the BRW in time $[\widetilde{t}_x, t_x]$:

- Look at a particle $v\in V_{\widetilde{t}_x}$ that is near the location $x - m_{(\log x)^2} - \ell$, $\ell\in\mathbb{Z}$.
- Consider the BRW process initiated at v. Condition on the *heterogeneity index* h of v, defined as the age of the *latest common ancestor* of all particles present in $[x,\infty)$ at time t_x, in the sub-tree initiated at v. For

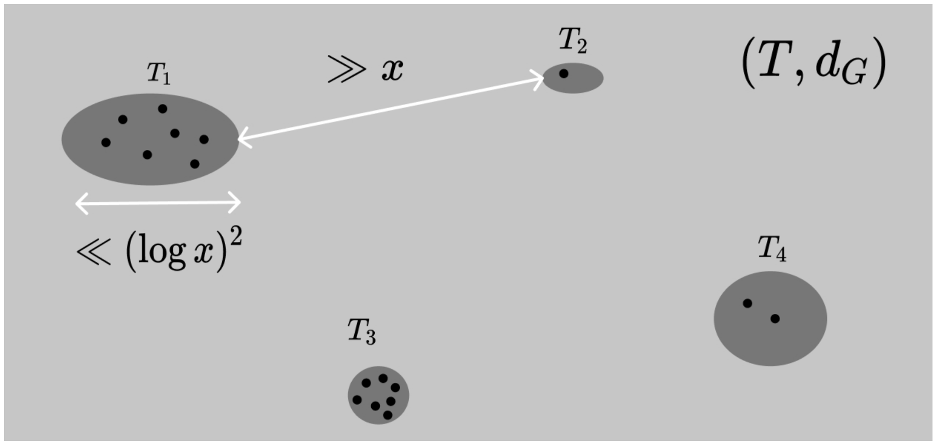

Figure 2: A typical cluster structure of the (random) set T. Consider the metric d_G on T defined as the genealogical distance of two particles $v, w \in T$ (i.e., if $u \in V_\ell$ is the latest common ancestor of $v, w \in V_{t_x}$, then $d_G(v, w) = t_x - \ell$). Solid ellipses indicate the clusters. The set T consists of $\asymp x^{\frac{d-1}{2}}$ clusters that are well-separated by distances of order $\gg x$; each of the clusters has diameter $\ll (\log x)^2$, in the metric space (T, d_G). The dispersion of a cluster T_i can be measured as its radius in the metric d_G. The set T_1 has a large cardinality and a large dispersion; T_2 has a small cardinality and a small dispersion; etc.

example, if only one descendant of v reaches $[x, \infty)$ at time t_x, then $h = 1$. If none reaches, then $h = 0$. Obviously, $h \in [0, (\log x)^2] \cap \mathbb{Z}$.

- Condition on the event that the location of the latest common ancestor at time $t_x - h$ is near $x - m_h + g$, $g \in \mathbb{Z}$.

Figure 3 below illustrates the three parameters ℓ, h, g.

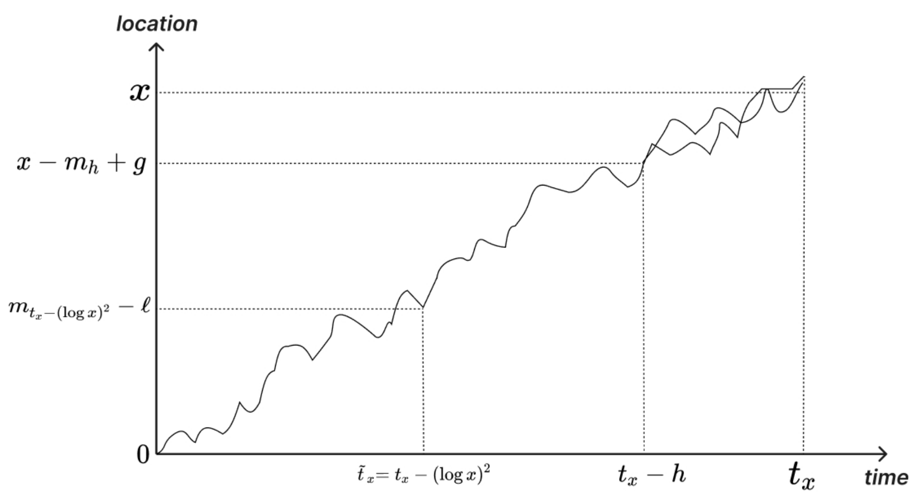

Figure 3: Illustration of the parameters ℓ, h, g. Solid curves indicate the trajectories of the BRW.

More precisely, we will apply a first moment method conditionally on ℓ, and apply a union bound on h, g. In this way, all particles $v \in V_{\tilde{t}_x}$ have been classified by the indices (ℓ, h, g). Following the above discussion, we first consider the following three cases:

(a) If ℓ is small,[8] the number of such particles v will be small. This will be shown in Proposition 10.

(b) If h is small, the size of the cluster corresponding to the particle v is small in the set T. This corresponds to the case (9) and will be proved in Lemma 13.

(c) If g is small, the number of descendants of v reaching x at time t_x is small (note that we condition on two particles separated at time $t_x - h$ that reach x at time t_x, so such a number must be positive), meaning that

[8] Here and below, the *smallness* of ℓ, g refers to having a (very) negative value, instead of having a small absolute value.

the cardinality of the cluster corresponding to v is small. This corresponds to the case (8) and will be proved in Lemma 29.

It remains to consider case (d): ℓ, h, g are all large. There are two sub-cases.

(d1) If ℓ, g are large and $h \approx (\log x)^2$, this means that in the small time period $[\widetilde{t}_x, t_x - h]$, the trajectory travels a distance of $\ell + g + m_{(\log x)^2} - m_h$ which is significantly larger than $m_{(\log x)^2 - h}$, and thus happens with a tiny probability. This is the goal of Lemma 19.

(d2) If ℓ, g are large and h is close to neither 0 nor $(\log x)^2$, we are in a situation where the BRW initiated by v satisfies that two descendants of v landing beyond $m_{(\log x)^2} + \ell$ in time $(\log x)^2$ have a common ancestor that is neither too early nor too late. This must happen rarely, since it contradicts the philosophy (7). The analysis is hidden in the computation of sums over h in the proofs of Lemmas 18 and 21.

Therefore, in all cases, the size of the cluster corresponding to the particle v can be controlled with high probability. One extra technicality comes into play since the statement (7) requires removing ballot-type events where the random walks cross a certain barrier. When applying (7) in case (d2), one needs to remove the barrier events for each $v \in V_{\widetilde{t}_x}$ within our consideration. Clearly, removing the events for all $v \in V_{\widetilde{t}_x}$ is extremely costly because there are exponentially many such particles. To overcome this issue, we remove barrier events only for those *relevant* $v \in V_{\widetilde{t}_x}$. These are the particles v where the last $(d-1)$-dimensional location of the latest common ancestor at time $t_x - h$ is close enough to the origin (say within a distance of $\asymp h$, so that it has a sufficient chance to reach B_x), in addition to satisfying the prescribed events. This finishes the upper bound for (6) and consequently the desired lower bound of τ_x.

Finally, we remark that in addition to nailing down the precise asymptotic of the FPT, our approach naturally leads to high probability properties of the trajectory that first realizes the FPT. We may identify the main contribution to the total size of the clusters emanating from distinct values of (ℓ, h, g)—it will become apparent from our proof that the main contribution stems from $\ell \asymp \log x, h = O(1)$, and $g = O(1)$. In other words, one can show that with high probability, the trajectory that realizes the FPT satisfies:

- its location at time $\widetilde{t}_x$ belongs to $[m_{t_x-(\log x)^2} - (\log x)/\varepsilon, m_{t_x-(\log x)^2} - \varepsilon \log x]$ for some small $\varepsilon > 0$;
- the collection of descendants of its ancestor at time $\widetilde{t}_x$ that reach $\mathbb{H}_x$ at time t_x has a latest common ancestor of age $O(1)$;
- if h denotes the age of that latest common ancestor, then its location at time $t_x - h$ is around $x - m_h + O(1)$.

Notation. We typically use (possibly with subscripts) u, v, w to denote particles; P, V, W to denote collections of particles; $E, G, H, \mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots, \mathscr{K}$ to denote events; t, τ, h to denote time; $x, g, \ell, \mathbf{x}, \mathbf{u}$ to denote locations or distances in Euclidean spaces; $\psi, \widehat{\psi}, \widetilde{\psi}, \overline{\psi}$ to denote barrier functions. Vectors are typically denoted by bold symbols. The notation $\delta > 0$ (resp. $L, C > 0$) typically refers to a small (resp. large) constant depending on the law of the BRW that may vary from line to line; $K_1, K_2, \ldots, K_{12}$ denote large constants that may depend on each other (in a permissible order) and the law of the BRW (including the underlying dimension d). Denote by $\mathbb{1}_A$ the indicator of an event A. The first time a definition appears is always followed by the ":=" sign. We refer to Appendix A for a glossary of frequently used notation and definitions throughout this paper.

Outline of the paper. Section 2 collects a few useful results on one-dimensional BRW and applies them to study the transition in the production number P_n, concluded by Section 2.3 that proves the desired upper bound of the FPT in Theorem 1. The proof of the corresponding lower bound takes up the entire Section 3, where we gradually carry out the multi-step conditioning plan outlined above. Section 4 contains a sketch of the extra arguments required for the proof of Theorem 2. Appendices B–D are devoted to several preliminary tools involving the escape probability of BRW, ballot theorems, and a conditional local CLT.

2 Preliminary results and proof of the upper bound of FPT

2.1 Useful results for the extremal behavior of one-dimensional BRW

This section contains a few useful lemmas that are *established* results for one-dimensional BRW. A few other results that need further verification will be collected in Appendix C. We assume throughout this section that the BRW satisfies assumptions (A1)–(A4) with $d = 1$, except for Lemma 7.

For $\beta > 0$ and $n \in \mathbb{N}$, we define the barrier event

$$\mathscr{G}_{n,\beta} := \bigcup_{v \in V_n} \bigcup_{0 \leqslant k \leqslant n} \left\{ \eta_{v,n}(k) \geqslant \frac{k m_n}{n} + \beta + \frac{6}{c_2} (\log \min\{k, n-k\})_+ \right\}, \tag{10}$$

where $(\cdot)_+$ denotes the positive part of an extended real number and by definition $(\log 0)_+ = (-\infty)_+ = 0$. Let us also define

$$\varphi_{n,\delta}(i) := e^{-\delta |i| \min(\frac{|i|}{n}, 1)}. \tag{11}$$

Lemma 3 (Lemma 2.4 of [9]). *There exists $\delta > 0$ such that*

$$\mathbb{P}(\mathscr{G}_{n,\beta}) \ll \beta e^{-c_2 \beta} \varphi_{n,\delta}(\beta). \tag{12}$$

Moreover, if $\tau_{n,\beta}$ denotes the smallest k such that the event $\mathscr{G}_{n,\beta}$ occurs, we have

$$\mathbb{P}(\tau_{n,\beta} = j) \ll \min\{j, n+1-j\}^{-3} \beta e^{-c_2 \beta} \varphi_{n,\delta}(\beta).$$

Proof. The first claim is precisely Lemma 2.4 of [9]. The second claim is a restatement of equation (23) therein, where the power is -3 instead of -2 because we changed the coefficient of the logarithm term in the definition (12) of $\mathscr{G}_{n,\beta}$. □

Lemma 4 (Corollary 2.5 and Lemma 2.7 of [9]). *There exist $\delta, C > 0$ such that for $z \geqslant 0$,*

$$\mathbb{P}(M_n > m_n + z) \leqslant C(z+1) e^{-c_2 z} \varphi_{n,\delta}(z). \tag{13}$$

Moreover, for $z \leqslant \sqrt{n}$,

$$\mathbb{P}(M_n > m_n + z) \geqslant \frac{1}{C} z e^{-c_2 z}. \tag{14}$$

In other words, (13) is tight up to constants for $z \leqslant \sqrt{n}$.

Remark 3. The proof of (14) proceeds by first applying the simple inequality

$$\mathbb{P}(M_n > m_n + z) \geqslant \mathbb{P}(\exists v \in V_n : \eta_{v,n}(n) \in [z, z+1)).$$

As a consequence, by slightly modifying the proof in [9], it holds that for $1 \leqslant z \leqslant \sqrt{n}$,

$$\mathbb{P}\Big(\exists v \in V_n, \Big| \eta_{v,n}(n) - (m_n + z) \Big| \leqslant \frac{1}{4} \Big) \geqslant \frac{1}{C} z e^{-c_2 z}. \tag{15}$$

Remark 4. The estimates (12) and (13) were stated in [9] in the form

$$\mathbb{P}(M_n > m_n + z) \leqslant C(z+1) e^{-c_2 z} e^{-\delta |i| \min(\frac{|i|}{n \log n}, 1)}.$$

On the other hand, the authors of [9] remarked below the statement of Lemma 2.4 therein that (13) holds with a slightly modified argument. We sketch the missing argument below for completeness. The only missing piece therein is the validity of equation (20), uniformly in $i < \beta - C\sqrt{n}$ instead of $i < \beta - C\sqrt{n \log n}$. After a proper change of measure using Lemma 2.2 therein, it suffices to show that for some $\delta > 0$, $\mathbb{P}(S_k < i) \ll e^{-\frac{\delta |i|^2}{n}}$ uniformly in $i \in [\beta - C\sqrt{n \log n}, \beta - C\sqrt{n}]$ and $1 \leqslant k \leqslant n$, where $S_k = \sum_{j=1}^k \xi_j$ is partial sum of an i.i.d. sequence with law given by the jump of the BRW. Using the Skorohod embedding theorem, we may write $S_k \overset{\text{law}}{=} B_{\tau_k}$ where B is Brownian motion and τ_k is a sum of k i.i.d. nonnegative random variables with a finite second moment. It then follows that uniformly for $1 \leqslant k \leqslant n$,

$$\mathbb{P}(S_k < i) = \mathbb{P}(B_{\tau_k} < i) \leqslant \mathbb{P}\Big(\tau_k \geqslant \frac{n}{4\delta}\Big) + \mathbb{P}\Big(\sup_{0 \leqslant s \leqslant n/(4\delta)} B_s > |i| \Big) \ll n^{-2} + e^{-\frac{|i|^2 4\delta}{2n}} \ll e^{-\frac{\delta |i|^2}{n}},$$

where we have used Remark 8.3 of [25] in the second inequality and the $\ll$ may depend on β, δ.

Define the collection of particles

$$Q_{n,\beta} := \left\{ v \in V_n : \text{ for any } 0 \leqslant k \leqslant n,\ \eta_{v,n}(k) < \frac{km_n}{n} + \beta + \frac{6}{c_2}(\log \min\{k, n-k\})_+ \right\}.$$

Lemma 5 (Proposition 9 of [8])**.** *Uniformly in $x \in [2, \sqrt{n}]$,*

$$\mathbb{E}[\#\{v \in Q_{n,\beta} : \eta_{v,n}(n) \geqslant m_n - x\}] \ll \beta(x+\beta)e^{c_2 x}.$$

Lemma 6 (Proposition 8 of [8])**.** *There exists $L > 0$ depending only on the law of the BRW such that the following holds conditioned upon survival. Given any $\varepsilon > 0$, there exists $C > 0$ independent from n and x such that uniformly for n large enough and for $x \in [2, \sqrt{n}]$,*

$$\mathbb{P}\left(\#\{v \in V_n : \eta_{v,n}(n) \geqslant m_n - x\} > Cxe^{c_2 x} \mid S\right) < \varepsilon \tag{16}$$

and

$$\mathbb{P}\left(\#\{v \in V_n : \eta_{v,n}(n) \geqslant m_n - x\} > \frac{1}{C}xe^{c_2 x} \mid S\right) > \frac{1}{L}. \tag{17}$$

The above results are closely related. For instance, Lemmas 3 and 5 together yield (16) as $\beta(x+\beta)e^{c_2 x} \asymp_\beta xe^{c_2 x}$. The proofs of Lemmas 5 and 6 are based on a "ballot theorem under a change of measure" argument, which is standard for the study of extrema of spatial branching processes and will be frequently used in this work. We refer to [8, 9] for further details.

The lower bound $1/L$ (instead of the anticipated stronger lower bound $1 - \varepsilon$) of the probability in (17) is an artifact of the second moment method. This bound solely does not suffice for proving high-probability upper bounds of τ_x. To resolve this issue, the work [8] established a concentration bound for τ_x around its median. Let $\mathrm{Med}(\cdot)$ denote the median of a random variable.

Lemma 7 (Theorem 2 of [8])**.** *Let τ_x be the first passage time to B_x for a d-dimensional BRW satisfying conditions (A1), (A2), and (A6). There exist constants $C, c > 0$ independent of x such that for each $y \in [0, x]$,*

$$\mathbb{P}\left(|\tau_x - \mathrm{Med}(\tau_x \mid S)| > y \mid S\right) \leqslant Ce^{-cy}.$$

2.2 Transition in the production number P_n

In this subsection, we formulate the quote (7) in the form we need using the notion of production numbers, keeping in mind that we look at a neighborhood of length $\log x$ near extrema. Recall (5) and that the production number P_n is defined as

$$P_n := \#\{v \in V_n : \exists w \in V_{t_x},\ w \succ v,\ \eta_{w,t_x}(t_x) \geqslant x\},\ 0 \leqslant n \leqslant t_x,$$

where $w \succ v$ means that particle w is a descendant of v. For our purpose, it is also useful to bound from below a similar quantity P'_n as P_n, defined as

$$P'_n := \#\left\{v \in V_n : \exists w \in V_{t_x},\ w \succ v,\ \eta_{w,t_x}(t_x) \in [x - \frac{1}{2}, x + \frac{1}{2}]\right\},\ 0 \leqslant n \leqslant t_x.$$

Here and later, the upper and lower limits of a sum are always interpreted as integers, without loss of generality. The main result in this subsection is the following extension of Lemma 6.

Proposition 8 (Transition in P_n)**.** *(i) For any $\varepsilon > 0$, there is $C > 0$ such that*

$$\mathbb{P}(P_{\tilde{t}_x} \geqslant Cx^{(d-1)/2} \mid S) < \varepsilon.$$

(ii) There are $L, C > 0$ such that

$$\mathbb{P}\left(P'_{(\log x)^2} \geqslant \frac{1}{C}x^{(d-1)/2} \mid S\right) > \frac{1}{L}.$$

Remark 5. A well-known fact of supercritical branching processes is that conditioned on extinction, the lifespan has an exponential tail. In particular, $\mathbb{P}(\#V_n > 0 \mid S^c) = o(1)$ (see Theorem 13.3 of [5]). It follows that the statement of Proposition 8 is essentially equivalent to the same statement without conditioning upon survival. The general idea behind proving Proposition 8 is to first condition on the configuration at the time of interest (say, $(\log x)^2$), classify the particles at such a time according to their locations (while discretizing the space), and finally evolve these particles independently until time t_x.

Proof. In the following, we use frequently the fact that

$$m_{\widetilde{t}_x} = x + \frac{d-1}{2c_2}\log x - c_1(\log x)^2 + o(1) = x - \Big(m_{(\log x)^2} - \frac{d-1}{2c_2}\log x + \frac{3}{c_2}\log\log x\Big) + o(1), \tag{18}$$

which follows from a direct computation.

(i) We first exclude a barrier event of arbitrarily small probability. Let $\varepsilon > 0$. By Lemma 3, $\mathbb{P}(Q_{n,\beta} \neq V_n) < \varepsilon/2$ for some β large enough. Therefore, we may without loss of generality assume that the event $\{Q_{n,\beta} = V_n\}$ holds. Define a collection of independent $\{0,1\}$-valued random variables $\{\delta_{v,y}\}_{v\in V_{\widetilde{t}_x},\, y\in[0,x]\cap\mathbb{N}}$, independent from everything else, and such that

$$\mathbb{P}(\delta_{v,y} = 1) = \mathbb{P}(M_{(\log x)^2} \geqslant x - y).$$

These random variables indicate whether a particle located within the interval $[y, y+1]$ at time $\widetilde{t}_x$ will have a descendant beyond x at time t_x (note that the evolution of the particles in time $[\widetilde{t}_x, t_x]$, given the configuration at time $\widetilde{t}_x$, are independent). For $u \in [2, \sqrt{n}]$ and $v \in V_n$, define the event

$$H_{v,n}(u) := \{\eta_{v,n}(n) \in [m_n - u, m_n - u + 1)\}.$$

Now on the event $\{Q_{n,\beta} = V_n\}$,

$$P_{\widetilde{t}_x} \preceq_{\mathrm{st}} \sum_{y=-\infty}^{m_{\widetilde{t}_x}+\beta} \sum_{v\in V_{\widetilde{t}_x}} \delta_{v,y} \mathbb{1}_{H_{v,\widetilde{t}_x}(m_{\widetilde{t}_x} - y)},$$

where $\preceq_{\mathrm{st}}$ denotes stochastic dominance. We then compute

$$\begin{aligned}
&\mathbb{E}\Bigg[\mathbb{1}_{\{Q_{n,\beta}=V_n\}} \sum_{y=-\infty}^{m_{\widetilde{t}_x}+\beta} \sum_{v\in V_{\widetilde{t}_x}} \delta_{v,y} \mathbb{1}_{H_{v,\widetilde{t}_x}(m_{\widetilde{t}_x} - y)}\Bigg] \\
&= \rho^{\widetilde{t}_x} \sum_{y=-\infty}^{m_{\widetilde{t}_x}+\beta} \mathbb{P}(M_{(\log x)^2} > x - y)\, \mathbb{P}(H_{v,\widetilde{t}_x}(m_{\widetilde{t}_x} - y) \cap \{Q_{n,\beta} = V_n\}).
\end{aligned}$$

Using the change of variable $j = m_{\widetilde{t}_x} - y$ and (18), the above is equal to

$$\begin{aligned}
\sum_{j=-\beta}^{\infty} \mathbb{P}\Big(M_{(\log x)^2} > m_{(\log x)^2} + (-\frac{d-1}{2c_2}\log x + j + \frac{3}{c_2}\log\log x)\Big) \\
\times \rho^{\widetilde{t}_x} \mathbb{P}(H_{v,\widetilde{t}_x}(j) \cap \{Q_{n,\beta} = V_n\}).
\end{aligned} \tag{19}$$

Let $C > 0$ be a large constant. We divide the sum over j in (19) into various ranges:

- $-\beta \leqslant j \leqslant \frac{d-1}{2c_2}\log x - \frac{3}{c_2}\log\log x$. By a union bound and a large deviation estimate, the total contribution is controlled by

$$\sum_{j=-\beta}^{\frac{d-1}{2c_2}\log x - \frac{3}{c_2}\log\log x} \rho^{\widetilde{t}_x} \mathbb{P}(H_{v,\widetilde{t}_x}(j)) \ll \sum_{j=-\beta}^{\frac{d-1}{2c_2}\log x - \frac{3}{c_2}\log\log x} (\log x)^3 e^{c_2 j} \ll x^{\frac{d-1}{2}}.$$

- $\frac{d-1}{2c_2}\log x - \frac{3}{c_2}\log\log x \leqslant j \leqslant C\log x$. By Lemmas 4 and 5, this part contributes at most

$$\begin{aligned}
&\sum_{j=\frac{d-1}{2c_2}\log x - \frac{3}{c_2}\log\log x}^{C\log x} \Big(-\frac{d-1}{2c_2}\log x + j + \frac{3}{c_2}\log\log x\Big) e^{-c_2(-\frac{d-1}{2c_2}\log x + j + \frac{3}{c_2}\log\log x)} j e^{c_2 j} \\
&\ll x^{\frac{d-1}{2}} (\log x)^{-3} \sum_{j=\frac{d-1}{2c_2}\log x - \frac{3}{c_2}\log\log x}^{C\log x} j\Big(-\frac{d-1}{2c_2}\log x + j + \frac{3}{c_2}\log\log x\Big) \\
&\ll x^{\frac{d-1}{2}}.
\end{aligned}$$

- $C\log x \leqslant j \leqslant \sqrt{x}$. Again applying Lemmas 4 and 5 leads to an upper bound of

$$x^{\frac{d-1}{2}}(\log x)^{-3}\sum_{j=C\log x}^{\infty} je^{-c_2 j-\frac{\delta j^2}{(\log x)^2}} je^{c_2 j}.$$

  The sum can be bounded using an integral approximation:

$$\sum_{j=C\log x}^{\infty} j\varphi_{(\log x)^2,\delta}(j)j \ll \int_{C\log x}^{(\log x)^2} y^2 e^{-\frac{\delta y^2}{(\log x)^2}}\mathrm{d}y + \int_{(\log x)^2}^{\infty} y^2 e^{-\delta y}\mathrm{d}y \ll (\log x)^3 \int_C^{\infty} z^2 e^{-\delta z}\mathrm{d}z \ll (\log x)^3.$$

  Therefore, this part of the contribution gives $\ll x^{\frac{d-1}{2}}$.

- $j \geqslant \sqrt{x}$. We directly apply the upper bound part of Cramér's theorem along with the first moment method. Using convexity of I, we obtain

$$\sum_{j=\sqrt{x}}^{\infty} \mathbb{P}\Big(M_{(\log x)^2} > m_{(\log x)^2} + j - \frac{d-1}{2c_2}\log x + \frac{3}{c_2}\log\log x\Big)\rho^{\widetilde{t}_x}\mathbb{P}(H_{v,\widetilde{t}_x}(j)\cap\{Q_{n,\beta}=V_n\})$$
$$\ll \sum_{j=\sqrt{x}}^{\infty} \rho^{t_x} e^{-(\log x)^2 I(\frac{m_{(\log x)^2}+j-\frac{d-1}{2c_2}\log x+\frac{3}{c_2}\log\log x}{(\log x)^2})} e^{-\widetilde{t}_x I(\frac{m_{\widetilde{t}_x}-j}{\widetilde{t}_x})}$$
$$\ll \sum_{j=\sqrt{x}}^{\infty} x^{\frac{d-1}{2}}\rho^{(\log x)^2} e^{-(\log x)^2 I(\frac{m_{(\log x)^2}+j-\frac{d-1}{2c_2}\log x+\frac{3}{c_2}\log\log x}{(\log x)^2})} e^{c_2 j}.$$

  If x is large enough, then for some $\delta,\delta'>0$,

$$\frac{m_{(\log x)^2}+j-\frac{d-1}{2c_2}\log x+\frac{3}{c_2}\log\log x}{(\log x)^2} \geqslant c_1+\delta' + \frac{(c_2+\delta)j}{I'(c_1+\delta')(\log x)^2},$$

  where we have used the fact that ϕ_ξ is well-defined in a neighborhood of c_2. This means

$$\sum_{j=\sqrt{x}}^{\infty} x^{\frac{d-1}{2}}\rho^{(\log x)^2} e^{-(\log x)^2 I(\frac{m_{(\log x)^2}+j-\frac{d-1}{2c_2}\log x+\frac{3}{c_2}\log\log x}{(\log x)^2})} e^{c_2 j} \ll x^{\frac{d-1}{2}} e^{(\log x)^2(I(c_1)-I(c_1+\delta'))}\sum_{j=\sqrt{x}}^{\infty} e^{-\delta j}$$
$$\ll x^{\frac{d-1}{2}}.$$

Combining the above four cases with (19), we conclude that

$$\mathbb{E}\Bigg[\mathbb{1}_{\{Q_{n,\beta}=V_n\}}\sum_{y=0}^{m_{\widetilde{t}_x}+\beta}\sum_{v\in V_{\widetilde{t}_x}}\delta_{v,y}\mathbb{1}_{H_{v,\widetilde{t}_x}(m_{\widetilde{t}_x}-y)}\Bigg] \ll x^{\frac{d-1}{2}}.$$

The rest follows from Markov's inequality and Remark 5.

(ii) Similarly as in (i), we define a collection of independent $\{0,1\}$-valued random variables $\{\delta'_{v,y}\}_{v\in V_{(\log x)^2},\, y\in[0,x]\cap\mathbb{N}}$, independent from everything else, and such that

$$\mathbb{P}(\delta'_{v,y}=1) = \mathbb{P}\Big(\exists v\in V_{\widetilde{t}_x},\ \eta_{v,\widetilde{t}_x}(\widetilde{t}_x)\in[x-y-\frac{1}{4}, x-y+\frac{1}{4}]\Big).$$

These random variables describe whether a particle located inside $[y-\frac{1}{4}, y+\frac{1}{4}]$ at time $(\log x)^2$ will end up with a descendant in $[x-\frac{1}{2}, x+\frac{1}{2}]$ at time t_x. Define

$$U_n := \Big\{v\in V_n:\ \text{for any } 0\leqslant k\leqslant n,\ \eta_{v,n}(k) < \frac{km_n}{n}\Big\}.$$

For $u\in[2,\sqrt{n}]$ and $v\in V_n$, define the event

$$H'_{v,n}(u) := \Big\{v\in U_n,\ \eta_{v,n}(n)\in[m_n-u-\frac{1}{4}, m_n-u+\frac{1}{4})\Big\}.$$

It follows that, by considering particles located in $[y-\frac{1}{4}, y+\frac{1}{4}]$ for $m_{(\log x)^2} - \log x \leqslant y \leqslant m_{(\log x)^2},\ y \in \mathbb{Z}$ at time $(\log x)^2$,

$$P'_{(\log x)^2} \succeq_{\mathrm{st}} \sum_{y=m_{(\log x)^2}-\frac{d-1}{c_2}\log x}^{m_{(\log x)^2}-\frac{d-1}{2c_2}\log x} \sum_{v\in V_{(\log x)^2}} \delta'_{v,y} \mathbb{1}_{H'_{v,(\log x)^2}(m_{(\log x)^2}-y)}. \tag{20}$$

We apply the second moment method to give a lower bound of the right-hand side of (20). Let us emphasize that the events $H'_{v,n}(u)$ and the random variables $\delta'_{v,y}$ are independent. We have

$$\begin{aligned}&\mathbb{E}\Big[\sum_{y=m_{(\log x)^2}-\frac{d-1}{c_2}\log x}^{m_{(\log x)^2}-\frac{d-1}{2c_2}\log x} \sum_{v\in V_{(\log x)^2}} \delta'_{v,y} \mathbb{1}_{H'_{v,(\log x)^2}(m_{(\log x)^2}-y)}\Big] \\ &= \rho^{(\log x)^2} \sum_{y=m_{(\log x)^2}-\frac{d-1}{c_2}\log x}^{m_{(\log x)^2}-\frac{d-1}{2c_2}\log x} \mathbb{P}\Big(\exists v\in V_{\widetilde{t}_x},\ \eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [x-y-\frac{1}{4}, x-y+\frac{1}{4}]\Big)\, \mathbb{P}(H'_{v,(\log x)^2}(m_{(\log x)^2}-y)).\end{aligned} \tag{21}$$

It follows from the same argument leading to (17) in [8] that for $u\in[2,\sqrt{n}]$, $\rho^n \mathbb{P}(H'_{v,n}(u)) \gg u e^{c_2 u}$. With a change of variable $j = m_{(\log x)^2} - y$ and applying (15) of Lemma 4 and (18), the quantity in (21) is equal to

$$\begin{aligned}&\sum_{j=\frac{d-1}{2c_2}\log x}^{\frac{d-1}{c_2}\log x} \mathbb{P}\Big(\Big|M_{\widetilde{t}_x} - (m_{\widetilde{t}_x} + (j + \frac{3}{c_2}\log\log x - \frac{d-1}{2c_2}\log x))\Big| \leqslant \frac{1}{4}\Big) \rho^{(\log x)^2} \mathbb{P}(H'_{v,(\log x)^2}(j)) \\ &\gg \sum_{j=\frac{d-1}{2c_2}\log x}^{\frac{d-1}{c_2}\log x} \Big((j+\frac{3}{c_2}\log\log x - \frac{d-1}{2c_2}\log x) e^{-c_2(j+\frac{3}{c_2}\log\log x - \frac{d-1}{2c_2}\log x)}\Big)(j e^{c_2 j}) \\ &\gg (\log x)^{-3} x^{\frac{d-1}{2}} \sum_{j=\frac{d-1}{2c_2}\log x}^{\frac{d-1}{c_2}\log x} (j - \frac{d-1}{2c_2}\log x) j \\ &\gg x^{\frac{d-1}{2}}.\end{aligned}$$

We next compute the second moment of the right-hand side of (20). Expanding the square leads to

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{y=m_{(\log x)^2}-\frac{d-1}{c_2}\log x}^{m_{(\log x)^2}-\frac{d-1}{2c_2}\log x} \sum_{v\in V_{(\log x)^2}} \delta'_{v,y} \mathbb{1}_{H'_{v,(\log x)^2}(m_{(\log x)^2}-y)}\Big)^2\Big] \\ &= \sum_{s=1}^{(\log x)^2} \rho^{(\log x)^2+s} \sum_{y=m_{(\log x)^2}-\frac{d-1}{c_2}\log x}^{m_{(\log x)^2}-\frac{d-1}{2c_2}\log x} \sum_{y'=m_{(\log x)^2}-\frac{d-1}{c_2}\log x}^{m_{(\log x)^2}-\frac{d-1}{2c_2}\log x} \mathbb{E}[\delta'_{v,y}\delta'_{w,y'}] \\ &\qquad \times \mathbb{P}(H'_{v,(\log x)^2}(m_{(\log x)^2}-y) \cap H'_{w,(\log x)^2}(m_{(\log x)^2}-y')),\end{aligned}$$

where v, w have genealogical distance equal to $2s$. Meanwhile, applying the same argument leading to (19) in [8] gives that

$$\begin{aligned}&\mathbb{P}(H'_{v,(\log x)^2}(m_{(\log x)^2}-y) \cap H'_{w,(\log x)^2}(m_{(\log x)^2}-y')) \\ &\qquad \ll (m_{(\log x)^2}-y)(m_{(\log x)^2}-y') e^{c_2((m_{(\log x)^2}-y)+(m_{(\log x)^2}-y'))}.\end{aligned}$$

Combining the above steps and applying (13) of Lemma 4 with the change of variables $j = m_{(\log x)^2} - y,\ j' =$

$m_{(\log x)^2} - y'$, we have

$$\mathbb{E}\Big[\Big(\sum_{y=m_{(\log x)^2}-\frac{d-1}{c_2}\log x}^{m_{(\log x)^2}-\frac{d-1}{2c_2}\log x}\sum_{v\in V_{(\log x)^2}}\delta'_{v,y}\mathbb{1}_{H'_{v,(\log x)^2}(m_{(\log x)^2}-y)}\Big)^2\Big]$$
$$\ll \sum_{j=\frac{d-1}{2c_2}\log x}^{\frac{d-1}{c_2}\log x}\sum_{j'=\frac{d-1}{2c_2}\log x}^{\frac{d-1}{c_2}\log x}\mathbb{P}(M_{\widetilde{t}_x} > m_{\widetilde{t}_x} + (j + \frac{3}{c_2}\log\log x - \frac{d-1}{2c_2}\log x))$$
$$\times\,\mathbb{P}(M_{\widetilde{t}_x} > m_{\widetilde{t}_x} + (j' + \frac{3}{c_2}\log\log x - \frac{d-1}{2c_2}\log x))jj'e^{c_2(j+j')}$$
$$\ll (\log x)^{-6}x^{d-1}\sum_{j=\frac{d-1}{2c_2}\log x}^{\frac{d-1}{c_2}\log x}\sum_{j'=\frac{d-1}{2c_2}\log x}^{\frac{d-1}{c_2}\log x}(j + \frac{3}{c_2}\log\log x - \frac{d-1}{2c_2}\log x)(j' + \frac{3}{c_2}\log\log x - \frac{d-1}{2c_2}\log x)jj'$$
$$\ll x^{d-1}.$$

We conclude with the Paley–Zygmund inequality that there exist $C, L > 0$ such that

$$\mathbb{P}\Big(P'_{(\log x)^2} \geqslant \frac{1}{C}x^{(d-1)/2}\Big) > \frac{1}{L}.$$

The proof is then complete in view of Remark 5. □

2.3 Proof of the upper bound of FPT

The idea is rather simple: given Proposition 8, we obtain $x^{\frac{d-1}{2}}/C$ many independent trajectories in the time period $[(\log x)^2, t_x]$ that lead to $[x-\frac{1}{2}, x+\frac{1}{2}]$ at time t_x. It remains to argue that each of them has roughly a chance of $x^{-\frac{d-1}{2}}$ to reach $B_{\mathbf{0}}(\frac{1}{2})$ in the last $d-1$ coordinates. To justify this claim, we need a conditional local central limit theorem, as we have already conditioned on the displacement in the first dimension of the trajectories. The proof will be deferred to Appendix D. Let $\{\boldsymbol{\xi}_i\}_{i\in\mathbb{N}}$ be an i.i.d. sequence of random vectors with the same law as $\boldsymbol{\xi}$ and consider its partial sum $\mathbf{S}_n = \sum_{i=1}^n \boldsymbol{\xi}_i$.

Lemma 9 (conditional local CLT)**.** *Fix a large constant $L > 0$. Uniformly for $\boldsymbol{\lambda}(x) = O((\log x)^L)$,*

$$\mathbb{P}\Big(\boldsymbol{\lambda}(x) + \mathbf{S}_{\widetilde{t}_x} \in [x-\frac{1}{2}, x+\frac{1}{2}] \times B_{\mathbf{0}}(\frac{1}{2}) \mid \lambda(x) + S_{\widetilde{t}_x} \in [x-\frac{1}{2}, x+\frac{1}{2}]\Big) \asymp x^{-\frac{d-1}{2}},$$

where $B_{\mathbf{0}}(\frac{1}{2})$ is the ball of radius $1/2$ centered at $\mathbf{0} \in \mathbb{R}^{d-1}$.

Proof of the upper bound of Theorem 1. On the event $\{P'_{(\log x)^2} \geqslant \frac{1}{C}x^{(d-1)/2}\} \cap S$, we may label particles $\{v_j\}_{1\leqslant j\leqslant x^{(d-1)/2}/C}$ at time $(\log x)^2$ that allow for descendants $\{w_j\}_{1\leqslant j\leqslant x^{(d-1)/2}/C}$ in $[x-\frac{1}{2}, x+\frac{1}{2}]$ at time t_x. By a union bound and a rough large deviation estimate (recalling that $\widehat{\boldsymbol{\eta}}_{v,n}(k) \in \mathbb{R}^{d-1}$ is the last $d-1$ coordinates of $\boldsymbol{\eta}_{v,n}(k)$),

$$\mathbb{P}(\exists j \in \{1, \ldots, x^{(d-1)/2}/C\},\ \big\|\widehat{\boldsymbol{\eta}}_{v_j,(\log x)^2}((\log x)^2)\big\| \geqslant (\log x)^3) \ll \rho^{(\log x)^2}e^{-\delta'(\log x)^3} = o(1)$$

for some $\delta' > 0$, and hence we may without loss of generality assume that

$$\big\|\widehat{\boldsymbol{\eta}}_{w_j,t_x}((\log x)^2)\big\| = \big\|\widehat{\boldsymbol{\eta}}_{v_j,(\log x)^2}((\log x)^2)\big\| \leqslant (\log x)^3$$

for all j. Let $\widetilde{\mathbb{P}}$ be the conditional law upon the above setting (i.e., on the event $\{P'_{(\log x)^2} \geqslant \frac{1}{C}x^{(d-1)/2}\} \cap S$, the configuration up to time $(\log x)^2$, and the event that $\eta_{w_j,t_x}(t_x) \in [x-\frac{1}{2}, x+\frac{1}{2}]$ and $\big\|\widehat{\boldsymbol{\eta}}_{w_j,t_x}((\log x)^2)\big\| \leqslant (\log x)^3$ for all j). By Lemma 9, uniformly in j,

$$\widetilde{\mathbb{P}}\Big(\widehat{\boldsymbol{\eta}}_{w_j,t_x}(t_x) - \widehat{\boldsymbol{\eta}}_{w_j,t_x}((\log x)^2) \in B_{\mathbf{0}}(\frac{1}{2}) - \widehat{\boldsymbol{\eta}}_{w_j,t_x}((\log x)^2)\Big) \gg x^{-\frac{d-1}{2}}. \tag{22}$$

To prove the upper bound of τ_x, we show that one of the descendants of these particles $\{v_j\}$ realizes the FPT with an asymptotically positive probability. It suffices then to consider the sub-event that for some j, $\boldsymbol{\eta}_{w_j,t_x}(t_x) \in$

$[x-\frac{1}{2},x+\frac{1}{2}]\times B_{\mathbf{0}}(\frac{1}{2})$. Note that under the law $\widetilde{\mathbb{P}}$, the random variables $\{\widehat{\boldsymbol{\eta}}_{w_j,t_x}(t_x)-\widehat{\boldsymbol{\eta}}_{w_j,t_x}((\log x)^2)\}_{1\leqslant j\leqslant x^{(d-1)/2}/C}$ are independent. Therefore, by (22),

$$
\begin{aligned}
&\mathbb{P}\Big(\tau_x \leqslant t_x \mid \{P'_{(\log x)^2} \geqslant \frac{1}{C}x^{(d-1)/2}\} \cap S\Big)\\
&\geqslant \widetilde{\mathbb{P}}\Big(\exists j,\ \widehat{\boldsymbol{\eta}}_{w_j,t_x}(t_x)-\widehat{\boldsymbol{\eta}}_{w_j,t_x}((\log x)^2) \in B_{\mathbf{0}}(\frac{1}{2})-\widehat{\boldsymbol{\eta}}_{w_j,t_x}((\log x)^2)\Big)\\
&= 1-\prod_{1\leqslant j\leqslant x^{(d-1)/2}/C}\Big(1-\widetilde{\mathbb{P}}\Big(\widehat{\boldsymbol{\eta}}_{w_j,t_x}(t_x)-\widehat{\boldsymbol{\eta}}_{w_j,t_x}((\log x)^2) \in B_{\mathbf{0}}(\frac{1}{2})-\widehat{\boldsymbol{\eta}}_{w_j,t_x}((\log x)^2)\Big)\Big)\\
&\gg 1-\prod_{1\leqslant j\leqslant x^{(d-1)/2}/C}(1-x^{-\frac{d-1}{2}}) \gg \frac{1}{C}.
\end{aligned}
$$

By Proposition 8,

$$
\mathbb{P}(\tau_x \leqslant t_x \mid S) \gg \frac{1}{LC}.
$$

Using Lemma 7, we conclude that for any $\varepsilon>0$, there exists $K>0$ such that

$$
\mathbb{P}(\tau_x \geqslant t_x + K \mid S) < \varepsilon.
$$

This completes the proof. □

3 Proof of the lower bound of FPT

Recall (5) and $\widetilde{t}_x = t_x-(\log x)^2$. Let us define also $\widetilde{x} := x - m_{(\log x)^2}$. This section aims to prove that $\tau_x \geqslant t_x - O_{\mathbb{P}}(1)$. That is, for a fixed $\varepsilon>0$, we find a lag time $K>0$ such that for x large enough,

$$
\mathbb{P}(\tau_x \leqslant t_x - K) < \varepsilon. \tag{23}
$$

In all asymptotic upper bounds below, the asymptotic constant does not depend on K. We use the short-hand notation $t_{x,K} := t_x - K$. We omit the conditioning on the survival event S for notational brevity in all probabilities and expectations below.

3.1 Reducing the proof to the analysis of particles with a fixed ancestor at time $\widetilde{t}_x$

3.1.1 The key conditioning step

To prove the lower bound of τ_x, we follow the strategy outlined in Section 1.2: classify the particles near frontier at time $\widetilde{t}_x$ according to the locations in the first dimension, and then analyze the chances that their descendants reach B_x at time $t_{x,K}=t_x-K$ (*local hitting probabilities*). Consequently, the total hitting probability of B_x can be bounded from above using a first moment method, by weighting the local hitting probabilities by the density of the particles at a location near the frontier. The following proposition computes the desired weights. Let K_2 be a large constant such that $\mathbb{P}(\mathscr{G}_{\widetilde{t}_x,K_2}) < \varepsilon/2$, by Lemma 3.

Proposition 10 (density of particles at time $\widetilde{t}_x$)**.** *It holds that for $\ell \leqslant x/\log x$,*

$$
\mathbb{E}\Big[\#\{v\in V_{\widetilde{t}_x} : \eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [\widetilde{x}-\ell-1,\widetilde{x}-\ell)\}\mathbb{1}_{\mathscr{G}^c_{\widetilde{t}_x,K_2}}\Big] \ll e^{c_2\ell}(\log x+\ell_+)x^{\frac{d-1}{2}}(\log x)^{-3}.
$$

For $\ell \geqslant x/\log x$, there exists $L>0$ such that

$$
\mathbb{E}\Big[\#\{v\in V_{\widetilde{t}_x} : \eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [\widetilde{x}-\ell-1,\widetilde{x}-\ell)\}\mathbb{1}_{\mathscr{G}^c_{\widetilde{t}_x,K_2}}\Big] \ll e^{c_2\ell}(\log x+\ell_+)x^{\frac{d-1}{2}}(\log x)^{-3}e^{L\ell(\log x)/x}.
$$

Here, we allow the constant in $\ll$ to depend on K_2.

Proof. We apply the "ballot theorem under a change of measure" argument similarly as done in the proof of (16) (see Proposition 8 of [8]). Before performing the change of measure, we introduce $\widehat{\lambda} = I'(m_{\widetilde{t}_x}/\widetilde{t}_x)$, which is the value of λ where the supremum of (1) is attained with $x = m_{\widetilde{t}_x}/\widetilde{t}_x$. By (15) of [9], it holds $0 \leqslant c_2 - \widehat{\lambda} \ll (\log x)/x$. Let $\mathbb{Q}$ be defined by

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} := e^{-\widehat{\lambda}(\eta_{v,\widetilde{t}_x}(\widetilde{t}_x)-m_{\widetilde{t}_x})-\widetilde{t}_x I(m_{\widetilde{t}_x}/\widetilde{t}_x)} \asymp (\widetilde{t}_x)^{3/2}\rho^{-\widetilde{t}_x} e^{-\widehat{\lambda}(\eta_{v,\widetilde{t}_x}(\widetilde{t}_x)-m_{\widetilde{t}_x})}. \tag{24}$$

It follows that under $\mathbb{Q}$, $\{\eta_{v,\widetilde{t}_x}(k) - km_{\widetilde{t}_x}/\widetilde{t}_x\}_{k=0,\dots,\widetilde{t}_x}$ is a mean zero random walk. The ending location of the random walk is around $\widetilde{x} - \ell$, which is at a distance

$$m_{\widetilde{t}_x} + K_2 - (\widetilde{x} - \ell) = \frac{d-1}{2c_2}\log x - \frac{3}{c_2}\log\log x + K_2 + \ell \tag{25}$$

below the barrier.

We therefore have, using Lemma 2.3 of [9], (24), and (25),

$$\begin{aligned}
&\mathbb{E}\big[\#\{v \in V_{\widetilde{t}_x} : \eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell)\}\mathbb{1}_{\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}}\big] \\
&\ll \rho^{\widetilde{t}_x}\mathbb{P}(\eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell) \cap \mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}) \\
&\ll (\widetilde{t}_x)^{3/2} e^{-\widehat{\lambda}(\widetilde{x}-\ell-m_{\widetilde{t}_x})}\mathbb{Q}(\eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell) \cap \mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}) \\
&\ll K_2\Big(1 + \Big(\frac{d-1}{2c_2}\log x - \frac{3}{c_2}\log\log x + K_2 + \ell\Big)_+\Big)e^{-c_2(\widetilde{x}-\ell-m_{\widetilde{t}_x})}e^{L\ell(\log x)/x} \\
&\ll e^{c_2\ell}(\log x + \ell_+)x^{\frac{d-1}{2}}(\log x)^{-3}e^{L\ell(\log x)/x}.
\end{aligned}$$

The last term $e^{L\ell(\log x)/x} \ll 1$ if $\ell \leqslant x/\log x$. This proves the claim. □

It remains to fix a particle $v \in V_{\widetilde{t}_x}$ such that $\eta_{v,\widetilde{t}_x}(\widetilde{t}_x) = [\widetilde{x}-\ell-1, \widetilde{x}-\ell)$ and $\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}$ hold (recall (12)), and bound the probability of finding a descendant $w \in V_{t_{x,K}}$ of v with $\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x$, which is the task of the next theorem. In the following, we use $\mathbb{Q}^\ell = \mathbb{Q}^{\ell,v}$ to denote the probability measure on the BRW restricted to the descendants of v (i.e., the sub-tree with root v and we implicitly recognize v as the common ancestor), conditioning on $\eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell)$ and $\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}$.[9] Let K_3, K_6, K_8 be large constants to be determined later in the proof. Recall (11). For future use, we consider a large constant L to be determined and define the auxiliary function

$$\Psi_{(\log x)^2,\delta}(\ell) := \begin{cases} \ell & \text{if } \ell < \frac{\log x}{L}; \\ (\log x)^{1/3} + \ell e^{-\frac{\delta\ell^2}{(\log x)^2}} & \text{if } \frac{\log x}{L} \leqslant \ell \leqslant L\log x\log\log x; \\ \varphi_{(\log x)^2,\delta}(\ell) & \text{if } \ell \geqslant L\log x\log\log x. \end{cases} \tag{26}$$

Next, we define the key quantity $I_{\ell,x}$ according to the range of ℓ as follows: if $\ell < -K_3\log\log x$,

$$I_{\ell,x} := (\log x)^{2(d-1)}x^{-\frac{d-1}{2}}.$$

If $-K_3\log\log x \leqslant \ell \leqslant K_6\log\log x$,

$$I_{\ell,x} := C(\varepsilon, K)(\log\log x)^{K_7}(\log x)x^{-\frac{d-1}{2}}e^{-c_2\ell}.$$

If $\ell > K_6\log\log x$,

$$\begin{aligned}
I_{\ell,x} := \;&\varepsilon x^{-\frac{d-1}{2}}e^{-c_2\ell}\ell\varphi_{(\log x)^2,\delta}(\ell) + x^{-\frac{d-1}{2}}\ell e^{-c_2\ell}\varphi_{(\log x)^2,\delta}(\ell)(K^{d-1-K_8} + \ell^{-K_8/2}(\log x)^{2d}) \\
&+ C(\varepsilon, K)e^{-(c_2+\delta/4)\ell}(\log x)^{3d}x^{-\frac{d-1}{2}} \\
&+ C(\varepsilon)e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\Psi_{(\log x)^2,\delta}(\ell)\big(e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}} + (\log\log x)^{d+3}\ell^{-1/8}e^{\frac{\log\ell\log\log x}{8\ell}}\big).
\end{aligned}$$

Here, $C(\varepsilon)$, $C(\varepsilon, K)$ are constants to be determined, which may depend also on the constants K_3, K_6, K_8 but not on x, ℓ.

Theorem 11 (first passage contributions of particles located in $[\widetilde{x}-\ell-1, \widetilde{x}-\ell) \times \mathbb{R}^{d-1}$ at time $\widetilde{t}_x$)**.** *Let $\varepsilon > 0$. Then there exist $C(\varepsilon), C(\varepsilon, K) > 0$ such that for all $\ell \in \mathbb{Z}$ and x large enough,*

$$\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x) \ll I_{\ell,x}.$$

Here, the implicit constant in $\ll$ may depend on K_3, K_6, K_8 but does not depend on ε or K.

The proof of Theorem 11 is deferred till later and takes up the majority of the rest of this paper. In the next subsection, we finish the proof of the lower bound of the first passage time τ_x assuming Theorem 11 holds.

[9]Note that the location $\boldsymbol{\eta}_{v,\widetilde{t}_x}(\widetilde{t}_x)$ is not independent from the event $\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}$.

3.1.2 Proof of the lower bound of FPT

Before proceeding to the proof, we need Lemma 22 below, which asserts that there exists $K_1 > 0$ such that

$$\mathbb{P}(\|\boldsymbol{\eta}_{v,n}(n)\| \geqslant 1 \text{ for all } v \in V_n) \leqslant K_1 e^{-\sqrt{n}/K_1}.$$

In other words, it is unlikely that all particles at generation n appear outside the unit ball for n large. In this way, we reduce the proof to finding an upper bound on $\mathbb{P}(\exists\, w \in V_{t_{x,K}},\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x)$.

Proof of the lower bound of Theorem 1. With Lemma 22, it suffices to bound from above the probability of finding a particle in B_x at time $t_{x,K}$ (because on the complement of such an event, Lemma 22 shows that the probability that the FPT is smaller decays at least quasi-exponentially; see e.g. the proof of Theorem 1 of [46]). The first step is to impose the global barrier constraint $\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}$(see (10) for its definition). We have

$$\mathbb{P}(\exists\, w \in V_{t_{x,K}},\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x) \leqslant \mathbb{P}(\mathscr{G}_{\widetilde{t}_x,K_2}) + \mathbb{P}(\exists\, w \in V_{t_{x,K}},\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}). \tag{27}$$

In the following, we use $w \succ v$ to denote that the particle w is a descendant of v. Using the first moment method, we have

$$\begin{aligned}
&\mathbb{P}(\exists\, w \in V_{t_{x,K}},\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}) \\
&= \mathbb{P}(\exists\, v \in V_{\widetilde{t}_x}, w \in V_{t_{x,K}}, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}) \\
&\leqslant \mathbb{E}\Bigg[\mathbb{1}_{\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}} \sum_{v\in V_{\widetilde{t}_x}} \mathbb{1}_{\{\exists\, w\in V_{t_{x,K}}, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x\}}\Bigg] \\
&= \sum_{\ell\in\mathbb{Z}} \mathbb{E}\Bigg[\mathbb{E}\Bigg[\mathbb{1}_{\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}} \sum_{v\in V_{\widetilde{t}_x}} \mathbb{1}_{\{\eta_{v,\widetilde{t}_x}(\widetilde{t}_x)\in[\widetilde{x}-\ell-1,\widetilde{x}-\ell)\}} \mathbb{1}_{\{\exists\, w\in V_{t_{x,K}}, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x\}} \\
&\qquad\qquad \mid \mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2},\, \{\eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell)\}\Bigg]\Bigg] \\
&= \sum_{\ell\in\mathbb{Z}} \mathbb{E}\Bigg[\#\{v \in V_{\widetilde{t}_x} : \eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell)\} \mathbb{1}_{\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}} \mathbb{Q}^{\ell}(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x)\Bigg].
\end{aligned}$$

Note that the inner probability is bounded by the deterministic term $I_{\ell,x}$ by Theorem 11. Applying also Proposition 10, we get

$$\begin{aligned}
\mathbb{P}(\exists\, w \in V_{t_{x,K}},\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}) &\ll \sum_{\ell\in\mathbb{Z}} \mathbb{E}\Big[\#\{v \in V_{\widetilde{t}_x} : \eta_{v,\widetilde{t}_x}(\widetilde{t}_x) \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell)\} \mathbb{1}_{\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}}\Big] I_{\ell,x} \\
&\ll \sum_{\ell\in\mathbb{Z}} (e^{c_2\ell}(\log x + \ell_+) x^{\frac{d-1}{2}} (\log x)^{-3} e^{L\ell(\log x)/x}) I_{\ell,x}.
\end{aligned}$$

Inserting the definition of $I_{\ell,x}$ yields the following upper bound on the above quantity:

$$\begin{aligned}
&\sum_{\ell<-K_3\log\log x} (e^{c_2\ell}(\log x + \ell_+) x^{\frac{d-1}{2}} (\log x)^{-3})((\log x)^{2(d-1)} x^{-\frac{d-1}{2}}) \\
&\quad + \sum_{-K_3\log\log x\leqslant\ell\leqslant K_6\log\log x} (e^{c_2\ell}(\log x + \ell_+) x^{\frac{d-1}{2}} (\log x)^{-3})(C(\varepsilon,K)(\log\log x)^{K_7}(\log x) x^{-\frac{d-1}{2}} e^{-c_2\ell}) \\
&\quad + \sum_{\ell>K_6\log\log x} (e^{c_2\ell}(\log x + \ell) x^{\frac{d-1}{2}} (\log x)^{-3} e^{L\ell(\log x)/x}) \\
&\qquad\quad \times \Big(\varepsilon x^{-\frac{d-1}{2}} e^{-c_2\ell} \ell \varphi_{(\log x)^2,\delta}(\ell) + x^{-\frac{d-1}{2}} \ell e^{-c_2\ell} \varphi_{(\log x)^2,\delta}(\ell) \big(K^{d-1-K_8} + \ell^{-K_8/2}(\log x)^{2d}\big) \\
&\qquad\quad + C(\varepsilon,K) e^{-(c_2+\delta/4)\ell} (\log x)^{3d} x^{-\frac{d-1}{2}} \\
&\qquad\quad + C(\varepsilon) e^{-c_1c_2K/4} x^{-\frac{d-1}{2}} e^{-c_2\ell} \Psi_{(\log x)^2,\delta}(\ell) \big(e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}} + (\log\log x)^{d+3} \ell^{-1/8} e^{\frac{\log\ell\log\log x}{8\ell}}\big) \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Our goal is to show that $I_1 + I_2 + I_3$ can be made arbitrarily small as $\varepsilon \to 0$, $K \to \infty$, and $x \to \infty$ (in order).

We next estimate the sums I_1, I_2, I_3. With $K_3 > 0$ picked large enough, the term I_1 can be controlled using

$$I_1 \ll \sum_{\ell < -K_3 \log\log x} e^{c_2 \ell} (\log x)^{2(d-2)} = o(1).$$

For I_2, we have

$$I_2 \ll C(\varepsilon, K)(\log\log x)^{K_7+1}(\log x)^{-2} = o(1),$$

where by convention, the $o(1)$ term converges to 0 as $x \to \infty$, with rate possibly depending on ε, K.

For I_3, we further decompose into two parts. First,

$$\begin{aligned}
&\sum_{\ell > K_6 \log\log x} (e^{c_2\ell}(\log x + \ell) x^{\frac{d-1}{2}} (\log x)^{-3} e^{L\ell(\log x)/x}) \\
&\qquad \times \Big(\varepsilon x^{-\frac{d-1}{2}} e^{-c_2\ell} \ell \varphi_{(\log x)^2,\delta}(\ell) + x^{-\frac{d-1}{2}} \ell e^{-c_2\ell} \varphi_{(\log x)^2,\delta}(\ell) \big(K^{d-1-K_8} + \ell^{-K_8/2} (\log x)^{2d} \big) \Big) \\
&= (\log x)^{-3} \sum_{\ell > K_6 \log\log x} \ell (\log x + \ell) \varphi_{(\log x)^2,\delta}(\ell) (\varepsilon + K^{d-1-K_8} + \ell^{-K_8/2} (\log x)^{2d}) e^{L\ell(\log x)/x}.
\end{aligned}$$

With K_8 picked large enough, the above is $\ll \varepsilon + o(1)$. Second, we have (the case $\ell \geqslant x/\log x$ being almost identical as above, we remove the term $e^{L\ell(\log x)/x}$ for brevity)

$$\begin{aligned}
&\sum_{\ell > K_6 \log\log x} (e^{c_2\ell}(\log x + \ell) x^{\frac{d-1}{2}} (\log x)^{-3}) \times \Big(C(\varepsilon, K) e^{-(c_2+\delta/4)\ell} (\log x)^{3d} x^{-\frac{d-1}{2}} \\
&\qquad + C(\varepsilon) e^{-c_1 c_2 K/4} x^{-\frac{d-1}{2}} e^{-c_2\ell} \Psi_{(\log x)^2,\delta}(\ell) \big(e^{2K_{10} \frac{\ell \log\log x}{(\log x)^2}} + (\log\log x)^{d+3} \ell^{-1/8} e^{\frac{\log \ell \log\log x}{8\ell}} \big) \Big) \\
&\ll C(\varepsilon, K)(\log x)^{3d-3} \sum_{\ell > K_6 \log\log x} e^{-\delta\ell/4} (\log x + \ell) \\
&\quad + C(\varepsilon)(\log x)^{-3} e^{-c_1 c_2 K/4} \sum_{\ell > K_6 \log\log x} (\log x + \ell) \big(e^{2K_{10} \frac{\ell \log\log x}{(\log x)^2}} + (\log\log x)^{d+3} \ell^{-1/8} e^{\frac{\log \ell \log\log x}{8\ell}} \big) \Psi_{(\log x)^2,\delta}(\ell) \\
&\ll C(\varepsilon, K)(\log x)^{3d-2-K_6\delta/5} \\
&\quad + C(\varepsilon)(\log x)^{-3} e^{-c_1 c_2 K/4} \Bigg(\sum_{\ell > K_6 \log\log x} (\log x + \ell) \Psi_{(\log x)^2,\delta}(\ell) e^{2K_{10} \frac{\ell \log\log x}{(\log x)^2}} \\
&\qquad + (\log\log x)^{d+3} \sum_{\ell > K_6 \log\log x} \ell^{-1/8} (\log x + \ell) \Psi_{(\log x)^2,\delta}(\ell) e^{\frac{\log \ell \log\log x}{8\ell}} \Bigg).
\end{aligned}$$

If K_6 is large enough, the first term can be controlled by $o(1)$. On the other hand, for the quantity inside the last bracket, we apply (26) and observe that $e^{2K_{10} \frac{\ell \log\log x}{(\log x)^2}} \ll 1$ for $\ell \ll \log x \log\log x$, $e^{\frac{\log \ell \log\log x}{8\ell}} \ll 1$ for $\ell \gg \log x$, and $e^{\frac{\log \ell \log\log x}{8\ell}} \ll \ell^{1/16}$ for $\ell > K_6 \log\log x$ and K_6 picked large enough. Using standard integral approximations and changes of variables, we have the following upper bound:

$$\begin{aligned}&\sum_{\ell>K_6\log\log x}(\log x+\ell)\Psi_{(\log x)^2,\delta}(\ell)e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}\\&\qquad+(\log\log x)^{d+3}\sum_{\ell>K_6\log\log x}\ell^{-1/8}(\log x+\ell)\Psi_{(\log x)^2,\delta}(\ell)e^{\frac{\log\ell\log\log x}{8\ell}}\\&\ll\int_{K_6\log\log x}^{(\log x)/L}(\log x+y)y\mathrm{d}y+\int_{(\log x)/L}^{L\log x\log\log x}\Big((\log x+y)(\log x)^{1/3}+y(\log x+y)e^{-\frac{\delta y^2}{(\log x)^2}}\Big)\,\mathrm{d}y\\&\qquad+\int_{L\log x\log\log x}^{\infty}\varphi_{(\log x)^2,\delta}(y)e^{2K_{10}\frac{y\log\log x}{(\log x)^2}}\,\mathrm{d}y\\&\qquad+(\log\log x)^{d+3}\Big(\int_{K_6\log\log x}^{(\log x)/L}(\log x+y)y^{7/8}e^{\frac{\log y\log\log x}{8y}}\,\mathrm{d}y+\int_{L\log x\log\log x}^{\infty}y^{-1/8}\varphi_{(\log x)^2,\delta}(y)\,\mathrm{d}y\\&\qquad+\int_{(\log x)/L}^{L\log x\log\log x}\Big((\log x+y)(\log x)^{1/3}y^{-1/8}+y^{7/8}(\log x+y)e^{-\frac{\delta y^2}{(\log x)^2}}\Big)\,\mathrm{d}y\Big)\\&\ll(\log x)^3+(\log x)^{8/3}+(\log\log x)^{d+3}(\log x)^{47/16}\\&\ll(\log x)^3.\end{aligned}$$

Combining the estimates above, we arrive at

$$I_3\ll o(1)+\varepsilon+C(\varepsilon)e^{-c_1c_2K/4}.$$

Altogether, we have

$$\mathbb{P}(\exists\, w\in V_{t_{x,K}},\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2})\ll I_1+I_2+I_3\ll o(1)+\varepsilon+C(\varepsilon)e^{-c_1c_2K/4}. \tag{28}$$

Recall that the $o(1)$ term may depend on ε, K but the implicit constant in $\ll$ does not depend on ε, K, x. Therefore, (28) can be made arbitrarily small by picking in order ε small enough, K large enough, and then $x\to\infty$. On the other hand, $\mathbb{P}(\mathscr{G}_{\widetilde{t}_x,K_2})$ can be bounded using Lemma 3. Combining with (27) shows the desired lower bound (23) of τ_x. □

3.2 Proof of Theorem 11

The goal of this section is to prove Theorem 11. In the titles of the subsections below, the cases I, II, and III respectively refer to the three cases: $\ell<-K_3\log\log x$, $-K_3\log\log x\leqslant\ell\leqslant K_6\log\log x$, and $\ell>K_6\log\log x$.

3.2.1 Setting up stages

Recall that the law $\mathbb{Q}^\ell=\mathbb{Q}^{\ell,v}$ denotes the probability on the BRW restricted to the descendants of v, conditioning on $\eta_{v,\widetilde{t}_x}(\widetilde{t}_x)=[\widetilde{x}-\ell-1,\widetilde{x}-\ell)$ and $\mathscr{G}^{\mathrm{c}}_{\widetilde{t}_x,K_2}$. Our goal is to give an upper bound for the local hitting probability

$$\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x) \tag{29}$$

for different ranges of ℓ, as a function of ℓ, x.

If we ignore the $d-1$ dimensions, this is exactly the probability that BRW reaches a distance $m_{(\log x)^2}+\ell$ in time $(\log x)^2$. When we add the extra dimensions, we need to condition on the extra event of the displacements along the first coordinate of v for the previous $\widetilde{t}_x$ steps. We first introduce the necessary settings required for the proof of Theorem 11.

For a particle $v\in V_{\widetilde{t}_x}$, we define the *heterogeneity index* h_v of v as the largest number of uncommon generations for two descendants of v that reach $\mathbb{H}_x$ at time $t_{x,K}$. In other words,

$$h_v=\max\{h\geqslant K:\,\exists\, v_1\in V_{t_x-h}, v_2,v_3\in V_{t_{x,K}}, v_2,v_3\succ v_1\succ v, \eta_{v_i,t_{x,K}}(t_{x,K})\geqslant x, i=2,3\}.$$

It follows that $h_v\geqslant K$ and there exists a unique *latest common ancestor* (lca) at level t_x-h_v of those who reach $\mathbb{H}_x$ at time $t_{x,K}$. We denote that latest common ancestor by v_{lca}. We also define the *heterogeneity location* g_v of v such that $\eta_{v_{\mathrm{lca}},t_x-h_v}(t_x-h_v)=x+g_v-m_{h_v-K}$, and the *multi-dimensional heterogeneity location* $\mathbf{u}_{v_{\mathrm{lca}}}\in\mathbb{R}^{d-1}$ of v

such that $\boldsymbol{\eta}_{v_{\rm lca},t_x-h_v}(t_x-h_v)=(x+g_v-m_{h_v-K},\mathbf{u}_{v_{\rm lca}})$. Let $\mathbf{u}_v\in\mathbb{R}^{d-1}$ denote the multi-dimensional location of v, i.e., $\boldsymbol{\eta}_{v,\widetilde{t}_x}(\widetilde{t}_x)=(\widetilde{x}-\ell,\mathbf{u}_v)$. The quantities $h_v, g_v, \mathbf{u}_v$ can all be viewed as random variables under $\mathbb{Q}^\ell$. For a vector $\mathbf{u}=(u_1,\dots,u_{d-1})\in\mathbb{R}^{d-1}$, we let $R_{\mathbf{u}}$ be the $(d-1)$-dimensional rectangle $[u_1,u_1+1)\times\cdots\times[u_{d-1},u_{d-1}+1)$. For $h=K,\dots,(\log x)^2$, $g\in\mathbb{Z}$, and $\mathbf{u}\in\mathbb{Z}^{d-1}$, we define the events $\mathscr{B}_h=\{h_v=h\}$, $\mathscr{C}_g=\{g_v\in[g,g+1)\}$, and $\mathscr{D}_{\mathbf{u}}=\{\mathbf{u}_{v_{\rm lca}}\in R_{\mathbf{u}}\}$, $\mathscr{H}_{\mathbf{u}}=\{\mathbf{u}_{v_{\rm lca}}-\mathbf{u}_v\in R_{\mathbf{u}}\}$. See Figure 3 for an illustration of these parameters.

Since we will partition the probability space into a union of events of the form $\mathscr{B}_h\cap\mathscr{C}_g$, it is essential to bound the probabilities of those events under $\mathbb{Q}^\ell$, which is given by the next result.

Lemma 12 (size of the event $\mathscr{B}_h\cap\mathscr{C}_g$)**.** *It holds that*

$$\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g)\ll\min\Big\{1,(|g+\ell|+1)e^{-c_2(g+\ell+c_1K/2)}\varphi_{(\log x)^2,\delta}(g+\ell)\Big\}\min\{1,((|g|+1)e^{c_2g})^2\}.$$

Proof. Let $\mathscr{J}_{h,g}$ denote the event that there exists $w\in V_{t_x-h}$, $w\succ v$, such that $\eta_{w,t_x-h}(t_x-h)\in[x-m_{h-K}+g-1,x-m_{h-K}+g)$. In particular, the displacement of the BRW initiated by v is at least (by concavity of the logarithm and assuming K is large enough)

$$m_{(\log x)^2}-m_{h-K}+\ell+g\geqslant m_{(\log x)^2-h}+\ell+g+\frac{c_1K}{2}.$$

Let $\mathscr{I}_{n,g}$ denote the event that there exist two descendants in time n running above m_n-g with a common ancestor only at time 0, i.e.,

$$\mathscr{I}_{n,g}:=\Big\{\exists\, v,w\in V_n,\ \mathrm{lca}(v,w)=\emptyset,\ \eta_{v,n}(n)\geqslant m_n-g,\ \eta_{w,n}(n)\geqslant m_n-g\Big\},\ n\in\mathbb{N},\tag{30}$$

where $\emptyset$ denotes the unique particle at time zero of the BRW. For the event $\mathscr{B}_h\cap\mathscr{C}_g$ to hold, the event $\mathscr{J}_{h,g}$ must hold and if $v_{\rm lca}$ denotes the latest common ancestor at time t_x-h, the sub-BRW with root $v_{\rm lca}$ satisfies the event $\mathscr{I}_{h-K,g}$. By independence of the process before and after time t_x-h, Lemma 4, and Lemma 28, we conclude that

$$\begin{aligned}\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g)&\leqslant\mathbb{Q}^\ell(\mathscr{J}_{h,g})\,\mathbb{P}(\mathscr{I}_{h-K,g})\\&\ll\min\Big\{1,(|g+\ell|+1)e^{-c_2(g+\ell+c_1K/2)}\varphi_{(\log x)^2,\delta}(g+\ell)\Big\}\min\{1,((|g|+1)e^{c_2g})^2\},\end{aligned}$$

where we have also used the fact that a probability is trivially bounded by one. □

3.2.2 A uniform conditional probability bound and local hitting probabilities, I

This section aims to prove the following result that serves as a general conditional bound for (29).

Lemma 13 (uniform conditional probability bound)**.** *It holds for $|\ell|\ll x^{1/3}$, $h=K,\dots,(\log x)^2$, and $g\geqslant 0$ that*

$$\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},w\succ v,\ \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x\mid\mathscr{B}_h\cap\mathscr{C}_g)\ll h^{d-1}x^{-\frac{d-1}{2}}.$$

An immediate consequence of Lemma 13 is that

$$\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},\ w\succ v,\ \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x)\ll(\log x)^{2(d-1)}x^{-\frac{d-1}{2}}.$$

In particular, the estimate in Theorem 11 for $\ell<-K_3\log\log x$ holds. We need a few preparations to prove Lemma 13.

Lemma 14 (uniform local upper limit theorem)**.** *Uniformly in $|\ell|\ll x^{1/3}$ and $\mathbf{u}\in\mathbb{R}^{d-1}$,*

$$\mathbb{Q}^\ell(\mathbf{u}_v\in R_{\mathbf{u}})\ll x^{-\frac{d-1}{2}}.$$

Proof. Recall that under the law $\mathbb{Q}^\ell$, we condition on $\eta_{v,\widetilde{t}_x}(\widetilde{t}_x)\in[\widetilde{x}-\ell-1,\widetilde{x}-\ell)$ and $\mathscr{G}^{\rm c}_{\widetilde{t}_x,K_2}$. In particular, the latter event means that the trajectory $\{\eta_{v,\widetilde{t}_x}(k)\}_{1\leqslant k\leqslant\widetilde{t}_x}$ is bounded from above by the curve

$$\psi(k):=\frac{km_{\widetilde{t}_x}}{\widetilde{t}_x}+K_2+\frac{6}{c_2}(\log\min\{k,\widetilde{t}_x-k\})_+,\ 1\leqslant k\leqslant\widetilde{t}_x.$$

Denote by $E_{\psi,x}$ the corresponding event that $S_k \leqslant \psi(k)$ for $1 \leqslant k \leqslant \widetilde{t}_x$. An equivalent formulation of the statement is that uniformly in $|\ell| \ll x^{1/3}$ and $\mathbf{u} \in \mathbb{R}^{d-1}$,

$$\mathbb{P}(\mathbf{S}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell) \times R_{\mathbf{u}} \mid S_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell),\, E_{\psi,x}) \ll x^{-\frac{d-1}{2}}.$$

The left-hand side probability can be written as

$$\mathbb{P}(\mathbf{S}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell) \times R_{\mathbf{u}} \mid S_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell),\, E_{\psi,x}) = \frac{\mathbb{P}(\mathbf{S}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell) \times R_{\mathbf{u}},\, E_{\psi,x})}{\mathbb{P}(S_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell),\, E_{\psi,x})}.$$

We next apply a change of measure argument similarly as in the proof of Proposition 10, whose notation we follow. Consider the measure $\mathbb{Q}$ defined by (24). It follows that under $\mathbb{Q}$, the jump $\boldsymbol{\xi}$ is i.i.d. with mean $(m_{\widetilde{t}_x}/\widetilde{t}_x, \mathbf{0}) \in \mathbb{R}^d$, and consequently, the random walk $\{\widetilde{\mathbf{S}}_n\} := \{\mathbf{S}_n - (m_{\widetilde{t}_x} n/\widetilde{t}_x, \mathbf{0})\}$ is centered. It follows that

$$\begin{aligned}
\frac{\mathbb{P}(\mathbf{S}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell) \times R_{\mathbf{u}},\, E_{\psi,x})}{\mathbb{P}(S_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell),\, E_{\psi,x})} &\asymp \frac{(\widetilde{t}_x)^{3/2} e^{-\widehat{\lambda}(\widetilde{x}-\ell-m_{\widetilde{t}_x})} \mathbb{Q}(\mathbf{S}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell) \times R_{\mathbf{u}},\, E_{\psi,x})}{(\widetilde{t}_x)^{3/2} e^{-\widehat{\lambda}(\widetilde{x}-\ell-m_{\widetilde{t}_x})} \mathbb{Q}(S_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell),\, E_{\psi,x})} \\
&= \frac{\mathbb{Q}(\widetilde{\mathbf{S}}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1-m_{\widetilde{t}_x}, \widetilde{x}-\ell-m_{\widetilde{t}_x}) \times R_{\mathbf{u}},\, E_{\psi,x})}{\mathbb{Q}(\widetilde{S}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1-m_{\widetilde{t}_x}, \widetilde{x}-\ell-m_{\widetilde{t}_x}),\, E_{\psi,x})},
\end{aligned}$$

where we note that

$$E_{\psi,x} = \{S_k \leqslant \psi(k) \text{ for all } 1 \leqslant k \leqslant \widetilde{t}_x\} = \Big\{\widetilde{S}_k \leqslant \psi(k) - \frac{k m_{\widetilde{t}_x}}{\widetilde{t}_x} \text{ for all } 1 \leqslant k \leqslant \widetilde{t}_x\Big\}.$$

Using (9) and (10) of Lemma 2.3 of [9], the denominator has the lower bound

$$\mathbb{Q}(\widetilde{S}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1-m_{\widetilde{t}_x}, \widetilde{x}-\ell-m_{\widetilde{t}_x}),\, E_{\psi,x}) \gg (\widetilde{x}-\ell-m_{\widetilde{t}_x}) x^{-3/2}.$$

Similarly, by Lemma 23,

$$\mathbb{Q}(\widetilde{\mathbf{S}}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1-m_{\widetilde{t}_x}, \widetilde{x}-\ell-m_{\widetilde{t}_x}) \times R_{\mathbf{u}},\, E_{\psi,x}) \ll (\widetilde{x}-\ell-m_{\widetilde{t}_x}) x^{-(d+2)/2}.$$

Note that here we allow the asymptotic constants to depend on K_2. Therefore,

$$\frac{\mathbb{P}(\mathbf{S}_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell) \times R_{\mathbf{u}},\, E_{\psi,x})}{\mathbb{P}(S_{\widetilde{t}_x} \in [\widetilde{x}-\ell-1, \widetilde{x}-\ell),\, E_{\psi,x})} \ll x^{-\frac{d-1}{2}},$$

as desired. ☐

Lemma 15 (uniform local upper limit theorem for an independent sum)**.** *For any $x > 0$, suppose that $\boldsymbol{\zeta}_1(x), \boldsymbol{\zeta}_2(x)$ are two independent random variables in $\mathbb{R}^{d-1}$ such that the law of $\boldsymbol{\zeta}_1(x)$ satisfies that for any $\mathbf{v} \in \mathbb{R}^{d-1}$,*

$$\mathbb{P}(\boldsymbol{\zeta}_1(x) \in R_{\mathbf{v}}) \ll x^{-\frac{d-1}{2}}.$$

Then for any $\mathbf{u} \in \mathbb{R}^{d-1}$,

$$\mathbb{P}(\boldsymbol{\zeta}_1(x) + \boldsymbol{\zeta}_2(x) \in R_{\mathbf{u}}) \ll x^{-\frac{d-1}{2}}.$$

Proof. It follows from independence and elementary geometry that

$$\begin{aligned}
\mathbb{P}(\boldsymbol{\zeta}_1(x) + \boldsymbol{\zeta}_2(x) \in R_{\mathbf{u}}) &= \sum_{\mathbf{v} \in \mathbb{R}^{d-1}} \mathbb{P}(\boldsymbol{\zeta}_1(x) \in R_{\mathbf{v}})\, \mathbb{P}(\boldsymbol{\zeta}_2(x) \in R_{\mathbf{u}} - R_{\mathbf{v}}) \\
&\ll x^{-\frac{d-1}{2}} \sum_{\mathbf{v} \in \mathbb{R}^{d-1}} \mathbb{P}(\boldsymbol{\zeta}_2(x) \in R_{\mathbf{u}} - R_{\mathbf{v}}) \ll x^{-\frac{d-1}{2}}.
\end{aligned}$$

This completes the proof. ☐

Lemma 16 (Markov property)**.** *It holds for all ℓ, h, g that under law $\mathbb{Q}^\ell$,*

$$\mathbf{u}_v \overset{\text{law}}{=} \mathbf{u}_v \mid \mathscr{B}_h, \mathscr{C}_g \overset{\text{law}}{=} \mathbf{u}_v \mid \mathscr{B}_h, \mathscr{C}_g, \sigma(\mathbf{u}_{v_{\text{lca}}} - \mathbf{u}_v).$$

Proof. By definition of the events, we have the following diagram of dependence:

$$\begin{array}{ccc} \sigma(\eta_{v,\widetilde{t}_x}(\widetilde{t}_x)) & \longrightarrow & \mathcal{F}^{(1)}_{\widetilde{t}_x,t_{x,K}} \\ \big\downarrow & & \big\downarrow \\ \sigma(\boldsymbol{\eta}_{v,\widetilde{t}_x}(\widetilde{t}_x)) & & \mathcal{F}^{(d-1)}_{\widetilde{t}_x,t_{x,K}} \end{array}$$

Figure 4: Dependence relation of the σ-algebras. Arrows indicate that under the probability measure induced by the BRW, any random variable that is measurable with respect to the object being pointed to (i.e., the head of the arrow) can be simulated as a function of random variables measurable with respect to the object pointing from (i.e., the tail of the arrow) along with some independent randomness. Here, $\mathcal{F}^{(1)}_{\widetilde{t}_x,t_{x,K}}$ denotes the σ-algebra generated by the first coordinate of the tree rooted at v, and $\mathcal{F}^{(d-1)}_{\widetilde{t}_x,t_{x,K}}$ denotes the σ-algebra generated by the last $d-1$ coordinates of the tree rooted at v.

Note that $\sigma(\eta_{v,\widetilde{t}_x}(\widetilde{t}_x))$ is trivial under $\mathbb{Q}^\ell$, $\mathbf{u}_v$ is $\sigma(\boldsymbol{\eta}_{v,\widetilde{t}_x}(\widetilde{t}_x))$-measurable, $\mathscr{B}_h \cap \mathscr{C}_g$ is $\mathcal{F}^{(1)}_{\widetilde{t}_x,t_{x,K}}$-measurable, and $\mathbf{u}_{v_{\text{lca}}} - \mathbf{u}_v$ is $\mathcal{F}^{(d-1)}_{\widetilde{t}_x,t_{x,K}}$-measurable. Therefore, on the law $\mathbb{Q}^\ell$, the random variable $\mathbf{u}_v$ is independent from $\mathscr{B}_h \cap \mathscr{C}_g$ and $\mathbf{u}_{v_{\text{lca}}} - \mathbf{u}_v$. This finishes the proof. ☐

Proof of Lemma 13. By the law of total probability,

$$\begin{aligned}
&\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}}, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x \mid \mathscr{B}_h \cap \mathscr{C}_g) \\
&= \sum_{\mathbf{u}} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}}, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x \mid \mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}})\, \mathbb{Q}^\ell(\mathscr{D}_{\mathbf{u}} \mid \mathscr{B}_h \cap \mathscr{C}_g).
\end{aligned}$$

Our first observation is that the conditioned law of the displacements of the tree initiated at u in the other $d-1$ dimensions is equivalent to its law conditioned only on the event that the same tree initiated at v_{lca} has at least two completely disjoint paths starting from v_{lca} that reach a distance of $m_{h-K} - g$ in time $h-K$ (which we denoted by $\mathscr{I}_{h-K,g}$ in (30)), since the rest of the conditioned events belong to other independent σ-algebras.[10] Since $g \geqslant 0$, the event $M^{(1)}_{h-K} > m_{h-K} - g$ is common, meaning that $\mathbb{P}(\mathscr{I}_{h-K,g}) \gg 1$, where the implicit constant may depend on K. By large deviation estimates for the maximum of a BRW (see Theorem 3.2 of [18] or Theorem 1.2 of [31]), we have for some $c, L > 0$ that

$$\begin{aligned}
&\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}}, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x \mid \mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}}) \\
&= \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}}, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x \mid \mathscr{I}_{h-K,g} \cap \mathscr{D}_{\mathbf{u}}) \\
&\ll \mathbb{P}(M^{(d-1)}_{h-K} \in R_{-\mathbf{u}} \mid M^{(1)}_{h-K} > m_{h-K} - g) \\
&\ll \min\{1, e^{-c(\|\mathbf{u}\| - Lh)}\}.
\end{aligned}$$

Here and later, the conditioned event $\mathscr{I}_{h-K,g}$ refers to the event that the sub-BRW process with root v_{lca} satisfies the event $\mathscr{I}_{h-K,g}$.

On the other hand, Lemma 16 implies that conditioning on $\mathscr{B}_h \cap \mathscr{C}_g$, the random variables $\mathbf{u}_v$ and $\mathbf{u}_{v_{\text{lca}}} - \mathbf{u}_v$ are independent under $\mathbb{Q}^\ell$. By Lemmas 14 and 15, and recalling the definition of $\mathscr{D}_{\mathbf{u}}$ above Lemma 12,

$$\mathbb{Q}^\ell(\mathscr{D}_{\mathbf{u}} \mid \mathscr{B}_h \cap \mathscr{C}_g) \ll x^{-\frac{d-1}{2}}. \tag{31}$$

Combining the above yields

$$\begin{aligned}
\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}}, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x \mid \mathscr{B}_h \cap \mathscr{C}_g) &\ll \sum_{\mathbf{u} \in \mathbb{R}^{d-1}} \min\{1, e^{-c(\|\mathbf{u}\| - Lh)}\} x^{-\frac{d-1}{2}} \\
&\ll h^{d-1} x^{-\frac{d-1}{2}},
\end{aligned}$$

as desired. ☐

[10] This fact will be frequently used below, such as in the proof of Lemma 18.

3.2.3 Local barrier events, II

We define a collection of *local* barrier events and show that they have small total probabilities. Consider for $\ell \geqslant -K_3 \log\log x$ the collection $W_{\ell,h,g}$ of particles $w \in V_{t_x-h}$, $w \succ v$ such that the event $\mathscr{B}_h \cap \mathscr{C}_g$ holds and $\|\widehat{\boldsymbol{\eta}}_{w,t_x-h}(t_x-h)\| \ll h$. Intuitively, these are the possible particles that can serve as the latest common ancestor v_{lca}. By Lemma 12 and (31), the number of such particles (under the global barrier event $\mathscr{G}^c_{\widetilde{t}_x,K_2}$) has an expectation

$$\mathbb{E}_{\mathbb{Q}^\ell}\left[\# W_{\ell,h,g} \mathbb{1}_{\mathscr{G}^c_{\widetilde{t}_x,K_2}}\right] \ll \min\left\{1, (|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\right\} \min\{1, ((|g|+1)e^{c_2 g})^2\} h^{d-1} x^{-\frac{d-1}{2}}. \quad (32)$$

For $w \in W_{\ell,h,g}$, we constrain the BRW initiated from w, in time $k \in [t_x-h, t_{x,K}]$, by the barrier

$$\widehat{\psi}_{g,h}(k) := x - m_{h-K} + g + L\log h + \frac{k-(t_x-h)}{h-K} m_{h-K} + \frac{4}{c_2}(\log \min\{k-(t_x-h), t_{x,K}-k\})_+, \quad t_x - h \leqslant k \leqslant t_{x,K}. \quad (33)$$

With L picked large enough, it follows from Lemma 3 that each local ballot probability is $\ll \varepsilon h^{-3d}$. This in particular means that L may depend on ε, and hence in the estimates below involving the barrier (33), the asymptotic constants may depend on ε. We state this dependence implicitly in Lemmas 18 and 21 but omit it in their proofs for simplicity. Using (32), the total local ballot probability then has an expectation

$$\begin{aligned}
&\sum_{h=K}^{(\log x)^2} \sum_{g\in\mathbb{Z}} (\varepsilon h^{-3d}) \min\left\{1, (|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\right\} \min\{1, ((|g|+1)e^{c_2 g})^2\} h^{d-1} x^{-\frac{d-1}{2}} \\
&\ll \varepsilon x^{-\frac{d-1}{2}} e^{-c_2\ell} \sum_{h=K}^{(\log x)^2} \sum_{g\in\mathbb{Z}} h^{-2d} (|g+\ell|+1) e^{-c_2 g} \varphi_{(\log x)^2,\delta}(g+\ell) \min\{1, ((|g|+1)e^{c_2 g})^2\} \\
&\ll \varepsilon x^{-\frac{d-1}{2}} e^{-c_2\ell} \sum_{g\in\mathbb{Z}} (|g+\ell|+1) e^{-c_2 g} \varphi_{(\log x)^2,\delta}(g+\ell) \min\{1, ((|g|+1)e^{c_2 g})^2\} \\
&\ll \varepsilon x^{-\frac{d-1}{2}} e^{-c_2\ell} \left(\sum_{g\geqslant 0} (|g+\ell|+1) e^{-c_2 g} \varphi_{(\log x)^2,\delta}(g+\ell) + \sum_{g<0} (|g+\ell|+1) e^{c_2 g} \varphi_{(\log x)^2,\delta}(g+\ell)(|g|+1)^2 \right) \\
&\ll \varepsilon x^{-\frac{d-1}{2}} e^{-c_2\ell} (|\ell|+1) \varphi_{(\log x)^2,\delta}(\ell).
\end{aligned}$$

As a summary, for the barrier event

$$\mathscr{E}_{1,\ell} := \bigcup_{K\leqslant h\leqslant(\log x)^2} \bigcup_{g\in\mathbb{Z}} \bigcup_{\substack{w\in V_{t_x-h} \\ w\in W_{\ell,h,g}}} \bigcup_{\substack{w'\in V_{t_{x,K}} \\ w'\succ w}} \bigcup_{t_x-h\leqslant k\leqslant t_{x,K}} \{\eta_{w',t_{x,K}}(k) > \widehat{\psi}_{g,h}(k)\}, \quad (34)$$

it holds that

$$\mathbb{Q}^\ell(\mathscr{E}_{1,\ell}) \ll \varepsilon x^{-\frac{d-1}{2}} e^{-c_2\ell} (|\ell|+1) \varphi_{(\log x)^2,\delta}(\ell). \quad (35)$$

Another local barrier event to be removed from our consideration is, roughly speaking, the random walk $\{\eta_{w,t_{x,K}}(k)\}_{\widetilde{t}_x\leqslant k\leqslant t_{x,K}}$ crosses a certain barrier *before* time t_x-h. Define for some large constant K_{11} (to be determined) the barrier function

$$\psi^*_{x,K}(k) := \widetilde{x} + K_{11}\log\log x + \frac{k-\widetilde{t}_x}{(\log x)^2} m_{(\log x)^2} + \frac{6}{c_2}(\log\min\{k-\widetilde{t}_x, t_{x,K}-k\})_+, \ \widetilde{t}_x \leqslant k \leqslant t_{x,K} \quad (36)$$

and the local barrier event

$$\mathscr{E}_{2,\ell} := \bigcup_{K\leqslant h\leqslant(\log x)^2} \left(\mathscr{B}_h \cap \Big(\bigcup_{u\in V_{t_x-h}} \Big(\bigcup_{\widetilde{t}_x\leqslant k\leqslant t_x-h} \{\eta_{u,t_x-h}(k) > \psi^*_{x,K}(k)\} \Big) \cap \Big(\bigcup_{\substack{w\in V_{t_{x,K}} \\ w\succ u}} \{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x\} \Big) \Big) \right). \quad (37)$$

To bound the size of $\mathscr{E}_{2,\ell}$, define

$$T^*_{\widetilde{t}_x} := \inf\left\{ k \in [\widetilde{t}_x, t_{x,K}] : \exists\, u \in V_k, u \succ v, \eta_{u,k}(k) \geqslant \psi_{x,K}(k) \right\}.$$

By Lemma 3 (with β therein given by $\ell + K_{11}\log\log x$ as the random walk starts from $[\widetilde{x}-\ell-1, \widetilde{x}-\ell)$),

$$\begin{aligned}\mathbb{Q}^\ell(T^*_{\widetilde{t}_x} = \widetilde{t}_x + j) &\ll \min\{j, (\log x)^2+1-j\}^{-3}(\ell + K_{11}\log\log x)e^{-c_2(\ell+K_{11}\log\log x)}\varphi_{(\log x)^2,\delta}(\ell) \\ &\ll \min\{j, (\log x)^2+1-j\}^{-3}(\ell + K_{11}\log\log x)e^{-c_2\ell}(\log x)^{-c_2K_{11}}.\end{aligned}$$

We may then compute the total contribution in the case where the barrier is crossed for a fixed ℓ. By Lemma 13 and since the barrier event is measurable at time $t_x - h$,

$$\begin{aligned}\mathbb{Q}^\ell(\mathscr{E}_{2,\ell}) &\leqslant \sum_{h=K}^{(\log x)^2} \mathbb{Q}^\ell\Big(\mathscr{B}_h \cap \{T^*_{\widetilde{t}_x} \leqslant t_x - h\} \cap \Big(\bigcup_{\substack{w\in V_{t_{x,K}} \\ w\succ u}} \{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x\}\Big)\Big) \\ &\ll \sum_{h=K}^{(\log x)^2} h^{d-1}x^{-\frac{d-1}{2}}(\ell + K_{11}\log\log x)e^{-c_2\ell}(\log x)^{-c_2K_{11}} \sum_{j=1}^{(\log x)^2-h} \min\{j,(\log x)^2+1-j\}^{-K_8} \\ &\ll x^{-\frac{d-1}{2}}(\ell + K_{11}\log\log x)e^{-c_2\ell}(\log x)^{2d-c_2K_{11}} \\ &\ll x^{-\frac{d-1}{2}}e^{-c_2\ell},\end{aligned} \tag{38}$$

where in the last step we picked K_{11} large enough and used that $|\ell| \ll \log\log x$. As a summary,

$$\mathbb{Q}^\ell(\mathscr{E}_{2,\ell}) \ll x^{-\frac{d-1}{2}}e^{-c_2\ell}. \tag{39}$$

3.2.4 Local hitting probabilities, II

In this section, we prove an upper bound of (29) in the case $-K_3\log\log x \leqslant \ell \leqslant K_6\log\log x$. Define

$$\mathscr{E}^*_\ell := \mathscr{E}_{1,\ell} \cup \mathscr{E}_{2,\ell} \tag{40}$$

and

$$\mathscr{K}^*_h := \mathscr{B}_h \cap \Big(\bigcup_{u\in V_{t_x-h}} \Big(\bigcup_{\widetilde{t}_x \leqslant k \leqslant t_x-h} \{\eta_{u,t_x-h}(k) > \psi^*_{x,K}(k)\}\Big) \cap \Big(\bigcup_{\substack{w\in V_{t_{x,K}} \\ w\succ u}} \{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x\}\Big)\Big).$$

Note that $\mathscr{E}_{2,\ell} = \bigcup_{K\leqslant h\leqslant(\log x)^2} \mathscr{K}_h$, so that $(\mathscr{E}^*_\ell)^{\mathrm{c}} \subseteq (\mathscr{K}^*_h)^{\mathrm{c}}$ for all $K \leqslant h \leqslant (\log x)^2$. Also define $k_h := (\log x)^2 - h$.

Lemma 17 (size of the event $\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}} \cap (\mathscr{K}^*_h)^{\mathrm{c}}$)**.** *Suppose that $-K_3\log\log x \leqslant \ell \leqslant K_6\log\log x$. It holds that uniformly for all $\mathbf{u} \in \mathbb{R}^{d-1}$, $K \leqslant h \leqslant (\log x)^2$, and $g \in \mathbb{Z}$,*

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}}) \ll x^{-\frac{d-1}{2}} \min\Big\{1, (|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\Big\} \min\{1, ((|g|+1)e^{c_2 g})^2\}. \tag{41}$$

Moreover, assume that $k_h \geqslant \log x$, and fix $K_5 > 0$. If $0 \leqslant g \leqslant K_5 \log h$,

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}} \cap (\mathscr{K}^*_h)^{\mathrm{c}}) \ll x^{-\frac{d-1}{2}} e^{-c_2(\ell+g)} \min\{k_h, h\}^{-5/4} (\log\log x)^2 e^{\frac{K_{10}(\log k_h) g}{k_h}}, \tag{42}$$

and if $g < 0$,

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}} \cap (\mathscr{K}^*_h)^{\mathrm{c}}) \ll x^{-\frac{d-1}{2}} e^{-c_2\ell} \min\{k_h, h\}^{-5/4} (\log\log x)^2 (|g|+1)^2 e^{c_2 g}. \tag{43}$$

Proof. First, we write

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}}) = \mathbb{Q}^\ell(\mathscr{D}_{\mathbf{u}} \mid \mathscr{B}_h \cap \mathscr{C}_g)\, \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g).$$

The first probability can be controlled by (31), and the second probability by Lemma 12. Inserting these estimates proves (41).

To prove (42) and (43), we apply the ballot theorem under a change of measure similarly as in the proof of Proposition 10, with the barrier given by (36). The starting location of the BRW at time $\widetilde{t}_x$ is $\widetilde{x} - \ell$, which is of distance $\ell + K_{11}\log\log x$ below the barrier $\psi^*_{x,K}(\widetilde{t}_x)$. The end location of the BRW at time $\widetilde{t}_x + k_h$ is $x - m_{h-K} + g$, which is of distance $O(\log\min\{k_h, h\}) - g - \frac{c_1 K}{2} + K_{11}\log\log x$ below the barrier $\psi^*_{x,K}(\widetilde{t}_x + k_h)$. Applying Lemma 2.3 of [9], if $0 \leqslant g \leqslant K_5 \log h$,

$$\begin{aligned}
&\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap (\mathscr{K}_h^*)^{\mathrm{c}}) \\
&\ll \rho^{k_h}(\rho^{-k_h} k_h^{3/2} e^{-\hat{\lambda}(x - m_{h-K} + g - (\widetilde{x} - \ell) - m_{k_h})})(k_h^{-3/2}(\ell + K_{11}\log\log x)(\log\min\{k_h, h\} + K_{11}\log\log x)) \\
&\ll e^{-c_2(\ell+g)} \min\{k_h, h\}^{-3/2}(\log\log x)^2 e^{(c_2 - \hat{\lambda})(m_{(\log x)^2} - m_{h-K} - m_{k_h} + \ell + g)}.
\end{aligned}$$

To further bound the term $e^{(c_2-\hat{\lambda})(m_{(\log x)^2} - m_{h-K} - m_{k_h} + \ell + g)}$, we recall from the proof of Proposition 10 that $c_2 - \hat{\lambda} \leqslant K_{10}(\log k_h)/k_h$ for some $K_{10} > 0$. Since $k_h \geqslant \log x \to \infty$ as $x \to \infty$, we may assume that $c_2 - \hat{\lambda} \leqslant c_2/6$ by letting x be large enough. In this case,

$$e^{(c_2-\hat{\lambda})(m_{(\log x)^2} - m_{h-K} - m_{k_h})} \leqslant e^{c_2(m_{(\log x)^2} - m_{h-K} - m_{k_h})/6} \ll \min\{k_h, h\}^{1/4}.$$

In addition, since $k_h \geqslant \log x$ and $|\ell| \ll \log\log x$, we have $e^{\frac{K_{10}(\log k_h)\ell}{k_h}} \ll 1$. These considerations altogether lead to

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap (\mathscr{K}_h^*)^{\mathrm{c}}) \ll e^{-c_2(\ell+g)} \min\{k_h, h\}^{-5/4}(\log\log x)^2 e^{\frac{K_{10}(\log k_h)(\ell+g)}{k_h}}.$$

On the other hand, it follows from the same independent sum argument leading to (31) that $\mathbb{Q}^\ell(\mathscr{D}_{\mathbf{u}} \mid \mathscr{B}_h \cap \mathscr{C}_g \cap (\mathscr{K}_h^*)^{\mathrm{c}}) \ll x^{-\frac{d-1}{2}}$. This proves (42).

If $g < 0$, we take advantage of the rare event that two independent descendants run distances $m_{h-K} - g$ for time $h - K$ (given by Lemma 28). Applying the same arguments as in Lemma 12 and using Lemma 2.3 of [9], we have

$$\begin{aligned}
&\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap (\mathscr{K}_h^*)^{\mathrm{c}}) \\
&\ll \rho^{k_h}\big(\rho^{-k_h} k_h^{3/2} e^{-\hat{\lambda}(x - m_{h-K} + g - (\widetilde{x} - \ell) - m_{k_h})}\big)(k_h^{-3/2}(\ell + K_{11}\log\log x)(\log\min\{k_h, h\} + K_{11}\log\log x))(|g|+1)^2 e^{c_2 g} \\
&\ll e^{-c_2\ell} \min\{k_h, h\}^{-3/2}(|g|+1)^2 e^{c_2 g} e^{(c_2-\hat{\lambda})(m_{(\log x)^2} - m_{h-K} - m_{k_h} + \ell + g)}.
\end{aligned}$$

The rest follows similarly as the case $g \geqslant 0$, and we note that $e^{\frac{K_{10}(\log k_h)g}{k_h}} \ll 1$ for $g < 0$. ☐

Lemma 18 (first passage contribution)**.** *For $-K_3 \log\log x \leqslant \ell \leqslant K_6 \log\log x$, it holds for some $K_7 \geqslant 0$ that*

$$\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}}, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}}) \ll C(\varepsilon, K)(\log\log x)^{K_7}(\log x)x^{-\frac{d-1}{2}}e^{-c_2\ell}. \tag{44}$$

Proof. In the following, the asymptotic constants in $\ll$ may depend on ε, K. We condition on $\mathscr{B}_h, \mathscr{C}_g, \mathscr{D}_{\mathbf{u}}$ and apply the law of total probability to write

$$\begin{aligned}
&\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}}, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}}) \\
&= \sum_{h=K}^{(\log x)^2} \sum_{g\in\mathbb{Z}} \sum_{\mathbf{u}\in\mathbb{Z}^{d-1}} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h)\, \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}}).
\end{aligned} \tag{45}$$

Let us consider a large constant $K_5 > 0$ to be determined, and separate into three cases depending on the values of g. In the following, the asymptotic constants may depend on ε.

Case (a): $g \geqslant K_5 \log h$. In this case, we do *not* condition on $\mathscr{D}_{\mathbf{u}}$, but directly apply Lemmas 12 and 13 to get

$$\begin{aligned}
&\sum_{h=K}^{(\log x)^2} \sum_{g \geqslant K_5 \log h} \sum_{\mathbf{u} \in \mathbb{Z}^{d-1}} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h)\, \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}}) \\
&= \sum_{h=K}^{(\log x)^2} \sum_{g \geqslant K_5 \log h} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{C}_g \cap \mathscr{B}_h)\, \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g) \\
&\ll \sum_{h=K}^{(\log x)^2} \sum_{g \geqslant K_5 \log h} (h^{d-1} x^{-\frac{d-1}{2}})(\min\Big\{1, (|g+\ell|+1) e^{-c_2(g+\ell+c_1K/2)} \varphi_{(\log x)^2,\delta}(g+\ell)\Big\} \min\{1, ((|g|+1)e^{c_2 g})^2\}) \\
&= \sum_{h=K}^{(\log x)^2} \sum_{g > K_5 \log h} \min\Big\{1, (|g+\ell|+1) e^{-c_2(g+\ell)} \varphi_{(\log x)^2,\delta}(g+\ell)\Big\} h^{d-1} x^{-\frac{d-1}{2}} \\
&\leqslant \sum_{h=K}^{(\log x)^2} (|K_5 \log h + \ell| + 1) \min\Big\{1, e^{-c_2(K_5 \log h + \ell)} \varphi_{(\log x)^2,\delta}(K_5 \log h + \ell)\Big\} h^{d-1} x^{-\frac{d-1}{2}} \\
&\ll (\log\log x) e^{-c_2 \ell} x^{-\frac{d-1}{2}} \varphi_{(\log x)^2,\delta}(\ell),
\end{aligned}$$

where in the last step we pick K_5 large enough.

Case (b): $0 \leqslant g < K_5 \log h$. We further split into two sub-cases. First, consider $\mathbf{u}$ such that $\|\mathbf{u}\| \leqslant \sqrt{h} \log h$. We first compute an upper bound for

$$\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h).$$

Conditioned on $\mathscr{B}_h \cap \mathscr{C}_g$, the event that $\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x$ holds implies $w \succ v_{\mathrm{lca}}$. Denote by $B_{\mathbf{z}}$ the unit ball centered at $\mathbf{z} \in \mathbb{R}^d$. By independence (see footnote 10),

$$\begin{aligned}
&\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h) \\
&\leqslant \sup_{\mathbf{u} \in \mathbb{R}^{d-1}} \mathbb{P}(\exists v \in V_{h-K},\, \boldsymbol{\eta}_{v,h-K}(h-K) \in B_{(m_{h-K}-g,\mathbf{u})},\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{I}_{h-K,g}) \\
&\ll \sup_{\mathbf{u} \in \mathbb{R}^{d-1}} \mathbb{P}(\exists v \in V_{h-K},\, \boldsymbol{\eta}_{v,h-K}(h-K) \in B_{(m_{h-K}-g,\mathbf{u})},\, (\mathscr{E}_\ell^*)^{\mathrm{c}}),
\end{aligned} \tag{46}$$

where the last step is because for $g \geqslant 0$, the event $\mathscr{I}_{h-K,g}$ that two descendants of v_{lca} separated at first step both reach $\mathbb{H}_x$ at time $t_{x,K}$ have a probability $\gg 1$, and hence we may remove the conditioning on $\mathscr{I}_{h-K,g}$ in (46) without changing the asymptotic upper bound. We have also abused notation by using $\mathscr{E}_{1,\ell}^{\mathrm{c}}$ to denote the event that the BRW is constrained by the barrier

$$k \mapsto L \log h + \frac{k}{h-K} m_{h-K} + \frac{4}{c_2}(\log \min\{k, h-K-k\})_+,\ 1 \leqslant k \leqslant h-K;$$

see (33). By Lemma 23 and a standard change of measure computation, we have uniformly in $\mathbf{u} \in \mathbb{R}^{d-1}$,

$$\mathbb{P}(\exists v \in V_{h-K},\, \boldsymbol{\eta}_{v,h-K}(h-K) \in B_{(m_{h-K}-g,\mathbf{u})},\, \mathscr{E}_{1,\ell}^{\mathrm{c}}) \ll g e^{c_2 g} (\log h)^2 (h-K+1)^{-\frac{d-1}{2}}.$$

Therefore, we arrive at

$$\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h) \ll g e^{c_2 g} (\log h)^2 (h-K+1)^{-\frac{d-1}{2}}. \tag{47}$$

Since the event $(\mathscr{K}_h^*)^{\mathrm{c}}$ depends only on times $[\widetilde{t}_x, \widetilde{t}_x + k_h]$ once we know a descendant of the latest common ancestor reaches B_x at time $t_{x,K}$, we obtain also that

$$\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h \cap (\mathscr{K}_h^*)^{\mathrm{c}}) \ll g e^{c_2 g} (\log h)^2 (h-K+1)^{-\frac{d-1}{2}}. \tag{48}$$

Fix $h \in [(\log x)^2 - \log x, (\log x)^2]$, i.e., $k_h \leqslant \log x$. Applying (47) and (41) of Lemma 17, we have

$$\begin{aligned}
&\sum_{g=0}^{K_5 \log h} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1} \\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h)\, \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}}) \\
&\ll \sum_{g=0}^{K_5 \log h} g e^{c_2 g} (\log h)^2 (h-K+1)^{-\frac{d-1}{2}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1} \\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}}) \\
&\ll \sum_{g=0}^{K_5 \log h} g e^{c_2 g} (\log h)^2 (h-K+1)^{-\frac{d-1}{2}} (\sqrt{h}\log h)^{d-1} x^{-\frac{d-1}{2}} \min\left\{1, (|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\right\} \\
&\ll \sum_{g=0}^{K_5 \log h} \min\left\{1, (|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\right\} g e^{c_2 g} (\log h)^{d+1} x^{-\frac{d-1}{2}} \left(\frac{h}{h-K+1}\right)^{\frac{d-1}{2}}.
\end{aligned}$$

The above quantity after summation over h is thus bounded by

$$\begin{aligned}
&\sum_{h=(\log x)^2-\log x}^{(\log x)^2} \sum_{g=0}^{K_5 \log h} \min\left\{1, (|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\right\} g e^{c_2 g} (\log h)^{d+1} x^{-\frac{d-1}{2}} \left(\frac{h}{h-K+1}\right)^{\frac{d-1}{2}} \\
&\ll (\log\log x)^{K_7} (\log x) x^{-\frac{d-1}{2}} \min\{e^{-c_2\ell}, 1\} \ll (\log\log x)^{K_7} (\log x) x^{-\frac{d-1}{2}} e^{-c_2\ell},
\end{aligned} \tag{49}$$

where we have used an integral approximation of a sum. Next, we consider $h \in [K, (\log x)^2 - \log x]$, i.e., $k_h \geqslant \log x$. Applying (48) and (42) of Lemma 17 and using that $h \leqslant (\log x)^2$, we have

$$\begin{aligned}
&\sum_{g=0}^{K_5 \log h} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1} \\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}_\ell^*)^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h \cap (\mathscr{K}_h^*)^{\mathrm{c}}) \\
&\qquad\qquad \times \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}} \cap (\mathscr{K}_h^*)^{\mathrm{c}}) \\
&\ll \sum_{g=0}^{K_5 \log h} g e^{c_2 g} (\log h)^2 (h-K+1)^{-\frac{d-1}{2}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1} \\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} x^{-\frac{d-1}{2}} e^{-c_2(\ell+g)} \min\{k_h, h\}^{-5/4} (\log\log x)^2 e^{\frac{K_{10}(\log k_h) g}{k_h}} \\
&\ll \min\{k_h, h\}^{-5/4} \left(\frac{h}{h-K+1}\right)^{\frac{d-1}{2}} x^{-\frac{d-1}{2}} e^{-c_2\ell} (\log h)^{d+1} (\log\log x)^2 \sum_{g=0}^{K_5 \log h} g e^{\frac{K_{10}(\log k_h) g}{k_h}} \\
&\ll \min\{k_h, h\}^{-5/4} x^{-\frac{d-1}{2}} e^{-c_2\ell} (\log\log x)^{K_{12}},
\end{aligned}$$

where in the second step, we used that $\#\{\mathbf{u} \in \mathbb{Z}^{d-1} : \|\mathbf{u}\| \leqslant \sqrt{h}\log h\} \ll h^{\frac{d-1}{2}} (\log h)^{d-1}$. Summing over h, we obtain

$$\sum_{h=K}^{(\log x)^2-\log x} \min\{k_h, h\}^{-5/4} x^{-\frac{d-1}{2}} e^{-c_2\ell} (\log\log x)^{K_{12}} \ll K^{\frac{d-1}{2}} x^{-\frac{d-1}{2}} e^{-c_2\ell} (\log\log x)^{K_{12}}. \tag{50}$$

Combining (49) and (50) yields a total contribution of at most

$$C(K)(\log\log x)^{K_7} (\log x) x^{-\frac{d-1}{2}} e^{-c_2\ell}.$$

Next, we consider $\mathbf{u}$ with $\|\mathbf{u}\| > \sqrt{h}\log h$. In this case, using a change of measure computation (without using ballot theorem) and a moderate deviation estimate (e.g., Theorem 3.7.1 of [14]), for some $\delta > 0$,

$$\begin{aligned}
&\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{E}_{1,\ell}^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h) \\
&\leqslant \sup_{\mathbf{u}' \in R_{\mathbf{u}}} \mathbb{P}(\exists v \in V_{h-K},\, \boldsymbol{\eta}_{v,h-K}(h-K) \in B_{(m_{h-K}-g,-\mathbf{u}')},\, \mathscr{E}_{1,\ell}^{\mathrm{c}}) \\
&\ll (h-K)^{3/2} e^{c_2 g} \mathbb{P}(\|\mathbf{S}_{h-K}\| \geqslant \|\mathbf{u}\|) \\
&\ll h^{3/2} e^{c_2 g} \varphi_{h,\delta}(\|\mathbf{u}\|).
\end{aligned}$$

Inserting into (45), we have

$$\begin{aligned}
&\sum_{h=K}^{(\log x)^2}\sum_{g=0}^{K_5\log h}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|>\sqrt{h}\log h}} \mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}^{\mathrm{c}}_{1,\ell}\mid \mathscr{D}_{\mathbf{u}}\cap\mathscr{C}_g\cap\mathscr{B}_h)\,\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{D}_{\mathbf{u}})\\
&\ll \sum_{h=K}^{(\log x)^2}\sum_{g=0}^{K_5\log h}\min\Big\{1,(|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\Big\}x^{-\frac{d-1}{2}}\sum_{k=\sqrt{h}\log h}^{\infty}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|\in[k,k+1]}} h^{3/2}e^{c_2 g}\varphi_{h,\delta}(\|\mathbf{u}\|)\\
&\ll \sum_{h=K}^{(\log x)^2}\sum_{g=0}^{K_5\log h}(|g+\ell|+1)e^{-c_2(g+\ell)}x^{-\frac{d-1}{2}}h^{3/2}e^{c_2 g}\sum_{k=\sqrt{h}\log h}^{\infty}k^{d-2}\varphi_{h,\delta}(k)\\
&\ll \sum_{h=K}^{(\log x)^2}\sum_{g=0}^{K_5\log h}(|g+\ell|+1)e^{-c_2\ell}x^{-\frac{d-1}{2}}h^{-100}\\
&\ll x^{-\frac{d-1}{2}}e^{-c_2\ell}(\log\log x)^2\varphi_{(\log x)^2,\delta}(\ell),
\end{aligned}$$

where we have used an integral approximation in the third step and that $|\ell|\ll\log\log x$ implies $\varphi_{(\log x)^2,\delta}(\ell)\gg 1$ in the last step.

Case (c): $g<0$. We exclude barrier events and compute the expected number of particles beyond x at time $t_{x,K}$ under the barrier event and conditioned on $\mathscr{C}_g$. Define the barrier event

$$\begin{aligned}
\mathscr{F}_{h,g} := \bigcup_{\substack{w\in V_{t_{x,K}}\\ w\succ v_{\mathrm{lca}}}}\ \bigcup_{0\leqslant k\leqslant h-K}\Big\{&\eta_{w,t_{x,K}}(\widetilde{t}_x-h+k)-\eta_{w,t_{x,K}}(\widetilde{t}_x-h)\\
&\geqslant L\log h - g + \frac{k}{h-K}m_{h-K}+\frac{4}{c_2}(\log\min\{k,h-K-k\})_+\Big\}.
\end{aligned}$$

for the sub-tree with root v_{lca}. Recall from (34) that on the event $\mathscr{E}^{\mathrm{c}}_{1,\ell}$, the event $\mathscr{F}_{h,g}$ cannot hold for each latest common ancestor v_{lca}.

In (45), the sum over $\mathbf{u}$ with $\|\mathbf{u}\|>\sqrt{h}\log h$ can be handled similarly as the case $g\geqslant 0$. By Lemma 29, we have the upper bound

$$\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}^{\mathrm{c}}_{1,\ell}\mid \mathscr{D}_{\mathbf{u}}\cap\mathscr{C}_g\cap\mathscr{B}_h)\ll h^{3/2}e^{-\frac{\delta\|\mathbf{u}\|^2}{h}}.$$

Inserting into (45), we have by Lemma 17 and arguing similarly in the case $g\geqslant 0$,

$$\begin{aligned}
&\sum_{h=K}^{(\log x)^2}\sum_{g<0}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|>\sqrt{h}\log h}} \mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}^{\mathrm{c}}_{1,\ell}\mid \mathscr{D}_{\mathbf{u}}\cap\mathscr{C}_g\cap\mathscr{B}_h)\,\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{D}_{\mathbf{u}})\\
&\ll \sum_{h=K}^{(\log x)^2}\sum_{g<0}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|>\sqrt{h}\log h}}\min\Big\{1,(|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\Big\}x^{-\frac{d-1}{2}}(|g|+1)^2e^{2c_2 g}h^{3/2}e^{-\frac{\delta\|\mathbf{u}\|^2}{h}}\\
&\ll \sum_{h=K}^{(\log x)^2}\sum_{g<0}\min\Big\{1,(|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\Big\}x^{-\frac{d-1}{2}}(|g|+1)^2e^{2c_2 g}h^{-100}\\
&\ll x^{-\frac{d-1}{2}}e^{-c_2\ell}\sum_{g<0}(|g+\ell|+1)(|g|+1)^2\varphi_{(\log x)^2,\delta}(g+\ell)e^{c_2 g}\\
&\ll x^{-\frac{d-1}{2}}e^{-c_2\ell}\varphi_{(\log x)^2,\delta}(\ell).
\end{aligned}$$

Note that, contrary to the case $g\geqslant 0$, here we do not have any constraint on the value of ℓ. The same computation will be re-used later in the proof of Lemma 21 when considering $\ell>K_6\log\log x$.

Let us now consider $\mathbf{u}$ with $\|\mathbf{u}\| \leqslant \sqrt{h}\log h$. Using independence (see footnote 10), the first probability on the right-hand side of (45) can be controlled by (similar consideration as the case $g \geqslant 0$)

$$\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{F}^{\mathrm{c}}_{h,g} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h)$$
$$\leqslant \sup_{\mathbf{u}' \in R_{\mathbf{u}}} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v_{\mathrm{lca}},\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) - \boldsymbol{\eta}_{w,t_{x,K}}(t_x - h) \in B_x - (x - m_{h-K} + g, \mathbf{u}'),\, \mathscr{F}^{\mathrm{c}}_{h,g} \mid \mathscr{I}_{h-K,g}).$$

Therefore, by Lemma 29,

$$\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{F}^{\mathrm{c}}_{h,g} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h) \ll (\log h)^2 (h-K+1)^{-\frac{d-1}{2}}. \tag{51}$$

Similarly as in (48), we also have

$$\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{F}^{\mathrm{c}}_{h,g} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h \cap (\mathscr{K}^*_h)^{\mathrm{c}}) \ll (\log h)^2 (h-K+1)^{-\frac{d-1}{2}}. \tag{52}$$

We then apply (51) and (41) of Lemma 17 to get for $(\log x)^2 - \log x \leqslant h \leqslant (\log x)^2$,

$$\sum_{h=(\log x)^2 - \log x}^{(\log x)^2} \sum_{g<0} \sum_{\substack{\mathbf{u} \in \mathbb{Z}^{d-1} \\ \|\mathbf{u}\| \leqslant \sqrt{h}\log h}} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{E}^{\mathrm{c}}_{1,\ell} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h)$$
$$\times\, \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}})$$
$$\ll \sum_{h=(\log x)^2 - \log x}^{(\log x)^2} \sum_{g<0} \sum_{\substack{\mathbf{u} \in \mathbb{Z}^{d-1} \\ \|\mathbf{u}\| \leqslant \sqrt{h}\log h}} (\log h)^2 (h-K+1)^{-\frac{d-1}{2}} x^{-\frac{d-1}{2}} |g+\ell| (|g|+1)^2 e^{c_2 g - c_2 \ell} \varphi_{(\log x)^2,\delta}(g+\ell)$$
$$\ll x^{-\frac{d-1}{2}} e^{-c_2\ell} \sum_{h=(\log x)^2 - \log x}^{(\log x)^2} \sum_{g<0} (\log h)^{d+1} |g+\ell| (|g|+1)^2 e^{c_2 g} \varphi_{(\log x)^2,\delta}(g+\ell) \Big(\frac{h}{h-K+1}\Big)^{\frac{d-1}{2}}$$
$$\ll x^{-\frac{d-1}{2}} (|\ell|+1) e^{-c_2\ell} (\log x)(\log\log x)^{d+4},$$

where in the second step, we used that $\#\{\mathbf{u} \in \mathbb{Z}^{d-1} : \|\mathbf{u}\| \leqslant \sqrt{h}\log h\} \ll h^{\frac{d-1}{2}} (\log h)^{d-1}$. For $K \leqslant h \leqslant (\log x)^2 - \log x$, we apply (52) and (43) of Lemma 17 to get

$$\sum_{h=K}^{(\log x)^2 - \log x} \sum_{g<0} \sum_{\substack{\mathbf{u} \in \mathbb{Z}^{d-1} \\ \|\mathbf{u}\| \leqslant \sqrt{h}\log h}} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, (\mathscr{E}^*_\ell)^{\mathrm{c}} \mid \mathscr{D}_{\mathbf{u}} \cap \mathscr{C}_g \cap \mathscr{B}_h \cap (\mathscr{K}^*_h)^{\mathrm{c}})$$
$$\times\, \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{D}_{\mathbf{u}} \cap (\mathscr{K}^*_h)^{\mathrm{c}})$$
$$\ll \sum_{h=K}^{(\log x)^2 - \log x} \sum_{g<0} \sum_{\substack{\mathbf{u} \in \mathbb{Z}^{d-1} \\ \|\mathbf{u}\| \leqslant \sqrt{h}\log h}} (\log h)^2 (h-K+1)^{-\frac{d-1}{2}} x^{-\frac{d-1}{2}} e^{-c_2\ell} \min\{k_h, h\}^{-5/4} (\log\log x)^2 (|g|+1)^2 e^{c_2 g} e^{\frac{K_{10}(\log k_h) g}{k_h}}$$
$$\ll \sum_{h=K}^{(\log x)^2 - \log x} \min\{k_h, h\}^{-5/4} K^{\frac{d-1}{2}} x^{-\frac{d-1}{2}} e^{-c_2\ell} (\log\log x)^{K_{12}}$$
$$\ll x^{-\frac{d-1}{2}} e^{-c_2\ell} (\log\log x)^{K_{12}}.$$

In total, we have a contribution of

$$C(K) x^{-\frac{d-1}{2}} e^{-c_2\ell} (\log x)(\log\log x)^{d+4}.$$

In summary, using that $-K_3 \log\log x \leqslant \ell \leqslant K_6 \log\log x$ implies $|\ell| \ll \log\log x$ and that $\varphi_{(\log x)^2,\delta}(\ell) \ll 1$, we conclude the following upper bound of (45):

$$C(K)(\log\log x)^{K_7} (\log x) x^{-\frac{d-1}{2}} e^{-c_2\ell}.$$

The proof is then complete. □

Remark 6. The reason we restrict to $\ell \leqslant K_6 \log\log x$ is that after multiplying the first passage contribution by the particle density from Proposition 10 and summing over ℓ, the $\log\log x$ power term in (44) will explode. In the following two subsections, we deal with the other case $\ell > K_6 \log\log x$.

3.2.5 Local barrier events, III

For the case $\ell > K_6 \log\log x$, we need to adjust the local barrier events in (40). Before this, we remove one more event that the heterogeneity index is close to $(\log x)^2$ and g is small simultaneously. Let $K_9 > 0$ be a large constant to be determined. Define the event

$$\mathscr{E}_{3,\ell} := \Big(\bigcup_{g \geqslant -\ell/K_4} \mathscr{C}_g\Big) \cap \Big(\bigcup_{(\log x)^2 - K_9\ell \leqslant h \leqslant (\log x)^2} \mathscr{B}_h\Big) = \Big\{g_v \geqslant -\frac{\ell}{K_4}\Big\} \cap \big\{h_v \geqslant (\log x)^2 - K_9 \ell\big\}, \tag{53}$$

where $K_4 > 0$ is a large constant to be determined.

Lemma 19 (Removing the event $\mathscr{E}_{3,\ell}$)**.** *It holds that for some $\delta > 0$ and all $\ell > K_6 \log\log x$,*

$$\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x, \mathscr{E}_{3,\ell}) \ll e^{-(c_2+\delta/4)\ell} (\log x)^{2(d-1)} x^{-\frac{d-1}{2}}.$$

Proof. We first need an improvement upon Lemma 12. For $(\log x)^2 - K_9\ell \leqslant h \leqslant (\log x)^2$, we have by independence that

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g) \leqslant \mathbb{P}(M_{(\log x)^2 - h} > g - m_{h-K} + m_{(\log x)^2} + \ell) \leqslant \mathbb{P}(M_{(\log x)^2 - h} > m_{(\log x)^2 - h} + g + \ell). \tag{54}$$

Since $(\log x)^2 - K_9\ell \leqslant h$, $g \geqslant -\ell/K_4$, and $\ell > K_6 \log\log x$, we have uniformly,

$$r_{g,h,\ell,x} := \frac{m_{(\log x)^2 - h} + g + \ell}{(\log x)^2 - h} \geqslant \frac{c_1 K_9 \ell - \frac{3}{c_2}\log\log x + (1 - K_4^{-1})\ell}{K_9 \ell} \geqslant c_1 + \delta_0$$

for some $\delta_0 > 0$ and all K_4, K_6 picked large enough (say, uniformly for all $K_4 > 2$ and $K_6 > 10/c_2$). Note that I is strictly convex in a neighborhood of c_1, which follows from Theorem 26.3 of [38] since (A4) implies that ξ has exponential moments in a neighborhood of c_2 and hence $\log \phi_\xi$ is smooth in a neighborhood of c_2. Consequently, $I(r_{g,h,\ell,x}) - I(c_1) \geqslant (c_2 + \delta_1)(r_{g,h,\ell,x} - c_1)$ for some $\delta_1 > 0$. Let us pick $\varepsilon_0 > 0$ small enough such that $(c_2 + \delta_1)(1 - \varepsilon_0) \geqslant c_2 + \delta$ for some $\delta > 0$. Then, with K_4, K_6 picked large enough depending on ε_0,

$$r_{g,h,\ell,x} = \frac{m_{(\log x)^2 - h} + g + \ell}{(\log x)^2 - h} \geqslant c_1 + \frac{g + \ell - \frac{3}{c_2}\log\log x}{(\log x)^2 - h} \geqslant c_1 + \frac{(1-\varepsilon_0)(g+\ell)}{(\log x)^2 - h}.$$

It follows that

$$I(r_{g,h,\ell,x}) - I(c_1) \geqslant (c_2 + \delta_1)(r_{g,h,\ell,x} - c_1) \geqslant \frac{(c_2+\delta)(g+\ell)}{(\log x)^2 - h}.$$

Consequently, by Cramér's large deviation upper bound and the union bound, for some $\delta > 0$,

$$\begin{aligned}\mathbb{P}(M_{(\log x)^2 - h} > m_{(\log x)^2 - h} + g + \ell) &\ll \rho^{(\log x)^2 - h}\, \mathbb{P}(S_{(\log x)^2 - h} > m_{(\log x)^2 - h} + g + \ell)\\ &\ll e^{-((\log x)^2 - h)(I(r_{g,h,\ell,x}) - I(c_1))} \leqslant e^{-(c_2+\delta)(g+\ell)}.\end{aligned}$$

By (54), we then arrive at

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g) \leqslant \mathbb{P}(M_{(\log x)^2 - h} > m_{(\log x)^2 - h} + g + \ell) \leqslant e^{-(c_2+\delta)(g+\ell)}. \tag{55}$$

By Lemma 13 and (55), with K_4 picked large enough,[11]

$$\begin{aligned}&\mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{E}_{3,\ell})\\ &\leqslant \sum_{h=(\log x)^2 - K_9\ell}^{(\log x)^2} \sum_{g \geqslant -\ell/K_4} \mathbb{Q}^\ell(\exists\, w \in V_{t_{x,K}},\, w \succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x,\, \mathscr{E}_{3,\ell} \mid \mathscr{C}_g \cap \mathscr{B}_h)\, \mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g)\\ &\ll \sum_{h=(\log x)^2 - K_9\ell}^{(\log x)^2} \sum_{g \geqslant -\ell/K_4} (\log x)^{2(d-1)} e^{-(c_2+\delta)(g+\ell)} x^{-\frac{d-1}{2}}\\ &\ll e^{-(c_2+\delta/4)\ell} (\log x)^{2(d-1)} x^{-\frac{d-1}{2}}.\end{aligned}$$

This finishes the proof. □

[11] Here we omit the case $|\ell| \gg x^{1/3}$, in which case the first passage probabilities decay exponentially in ℓ if $\ell \gg x^{1/3}$, and can be trivially bounded by 1 if $\ell \ll -x^{1/3}$.

On the event $\mathscr{E}_{3,\ell}^{\mathrm{c}}$, we may adjust the barrier event $\mathscr{E}_{2,\ell}$ as follows. Define for some large constant K_8 the following barrier function

$$\psi_{x,K}(k) := \widetilde{x} + \frac{k-\widetilde{t}_x}{(\log x)^2} m_{(\log x)^2} + \frac{2K_8}{c_2}(\log \min\{k-\widetilde{t}_x, t_{x,K}-k\})_+,\ \widetilde{t}_x \leqslant k \leqslant t_{x,K} \tag{56}$$

and the local barrier event

$$\mathscr{E}_{4,\ell} := \bigcup_{K\leqslant h\leqslant (\log x)^2-K_9\ell} \left(\mathscr{B}_h \cap \Big(\bigcup_{u\in V_{t_x-h}} \Big(\bigcup_{\widetilde{t}_x\leqslant k\leqslant t_x-h} \{\eta_{u,t_x-h}(k) > \psi_{x,K}(k)\}\Big) \cap \Big(\bigcup_{\substack{w\in V_{t_{x,K}}\\ w\succ u}} \{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x\}\Big)\Big)\right). \tag{57}$$

The following considerations are similar to Section 3.2.3, with a different range of ℓ. To bound the size of $\mathscr{E}_{4,\ell}$, define

$$T_{\widetilde{t}_x} := \inf\Big\{k \in [\widetilde{t}_x, t_{x,K}] : \exists\, u \in V_k, u \succ v, \eta_{u,k}(k) \geqslant \psi_{x,K}(k)\Big\}.$$

By Lemma 3 (with β therein given by ℓ as the random walk starts from $[\widetilde{x}-\ell-1, \widetilde{x}-\ell)$, and coefficient of log replaced by $2K_8/c_2$),

$$\mathbb{Q}^\ell(T_{\widetilde{t}_x} = \widetilde{t}_x + j) \ll \min\{j, (\log x)^2+1-j\}^{-K_8} \ell e^{-c_2\ell} \varphi_{(\log x)^2,\delta}(\ell).$$

We may then calculate the total contribution in the case where the barrier is crossed for a fixed ℓ, on the event $g \geqslant -\ell/K_4$ (and hence $h \leqslant (\log x)^2 - K_9\ell$ since we excluded the event $\mathscr{E}_{3,\ell}$). By Lemma 13 and since the barrier event is measurable at time $t_x - h$,

$$\begin{aligned}
\mathbb{Q}^\ell(\mathscr{E}_{4,\ell} \cap \mathscr{E}_{3,\ell}^{\mathrm{c}}) &\leqslant \sum_{h=K}^{(\log x)^2-K_9\ell} \mathbb{Q}^\ell\left(\mathscr{B}_h \cap \{T_{\widetilde{t}_x} \leqslant t_x - h\} \cap \Big(\bigcup_{\substack{w\in V_{t_{x,K}}\\ w\succ u}} \{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x\}\Big)\right) \\
&\ll \sum_{h=K}^{(\log x)^2-K_9\ell} h^{d-1} x^{-\frac{d-1}{2}} \ell e^{-c_2\ell} \varphi_{(\log x)^2,\delta}(\ell) \sum_{j=K_9\ell}^{(\log x)^2-h} \min\{j, (\log x)^2+1-j\}^{-K_8} \\
&\ll x^{-\frac{d-1}{2}} \ell e^{-c_2\ell} \varphi_{(\log x)^2,\delta}(\ell) \sum_{h=K}^{(\log x)^2-K_9\ell} h^{d-1} \max\{\ell^{-K_8/2}, h^{-K_8}\} \\
&\ll x^{-\frac{d-1}{2}} \ell e^{-c_2\ell} \varphi_{(\log x)^2,\delta}(\ell) \big(K^{d-1-K_8} + \ell^{-K_8/2} (\log x)^{2d}\big).
\end{aligned} \tag{58}$$

3.2.6 Local hitting probabilities, III

In this section, we prove an upper bound of (29) in the case $\ell > K_6 \log\log x$. The main improvement compared to Lemma 18 stems from an improvement of Lemma 17, after removing the (unlikely) barrier event $\mathscr{E}_{4,\ell}$ up to time $t_x - h$ defined in (57). Here and later, we denote by

$$\mathscr{E}_\ell := \mathscr{E}_{1,\ell} \cup \mathscr{E}_{3,\ell} \cup \mathscr{E}_{4,\ell} \tag{59}$$

and

$$\mathscr{K}_h := \mathscr{B}_h \cap \Big(\bigcup_{u\in V_{t_x-h}} \Big(\bigcup_{\widetilde{t}_x\leqslant k\leqslant t_x-h} \{\eta_{u,t_x-h}(k) > \psi_{x,K}(k)\}\Big) \cap \Big(\bigcup_{\substack{w\in V_{t_{x,K}}\\ w\succ u}} \{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x\}\Big)\Big).$$

Note that $\mathscr{E}_{4,\ell} = \bigcup_{K\leqslant h\leqslant (\log x)^2-K_9\ell} \mathscr{K}_h$, so that $\mathscr{E}_\ell^{\mathrm{c}} \subseteq \mathscr{K}_h^{\mathrm{c}}$ for all $k \leqslant h \leqslant (\log x)^2 - K_9\ell$.

By the same independent sum argument leading to (31), it is not hard to see that

$$\mathbb{Q}^\ell(\mathscr{D}_{\mathbf{u}} \mid \mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{K}_h^{\mathrm{c}}) \ll x^{-\frac{d-1}{2}}. \tag{60}$$

In view of the upper bounds in Lemmas 26 and 27 below (which we will apply with $\varepsilon = 1/3$), we fix a large constant $L > 0$ and define the following auxiliary function

$$\Phi_{n,\delta}(x,y) := \begin{cases} xyn^{-3/2} & \text{if } 0 \leqslant y < \frac{\sqrt{n}}{L}; \\ \min\{xn^{-4/3} + xyn^{-3/2} e^{-\frac{\delta y^2}{n}}, \varphi_{n,\delta}(y)\} & \text{if } y \geqslant \frac{\sqrt{n}}{L}, \end{cases} \tag{61}$$

where $x \in [0, O(n^{1/6})]$. The function $\Phi_{n,\delta}(x,y)$ serves as asymptotic upper bounds of ballot probabilities. Recall also the short-hand notation $k_h = (\log x)^2 - h$.

Lemma 20 (size of the event $\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{K}_h^{\mathrm{c}}$)**.** *Assume that* $\ell > K_6 \log\log x$*,* $k_h = (\log x)^2 - h \geqslant K_9 \ell$*, and* $g \geqslant \max\{-\ell/K_4, -k_h^{1/6}\}$*. If* $g \geqslant 0$*,*

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{K}_h^{\mathrm{c}}) \ll e^{-c_2(\ell+g)} \min\{k_h, h\}^{-5/4} k_h^{3/2} e^{-c_1 c_2 K/2} \Phi_{k_h,\delta}((\log \min\{k_h, h\} - g)_+, \ell) e^{\frac{K_{10}(\log k_h)(\ell+g)}{k_h}}. \quad (62)$$

If $g < 0$*,*

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{K}_h^{\mathrm{c}}) \ll e^{-c_2\ell} \min\{k_h, h\}^{-5/4} k_h^{3/2} (|g|+1)^2 e^{c_2 g} e^{-c_1 c_2 K/2} \times \Phi_{k_h,\delta}((\log \min\{k_h, h\} - g)_+, \ell) e^{\frac{K_{10}(\log k_h)(\ell+g)}{k_h}}. \quad (63)$$

On the other hand, in the case where $-\ell/K_4 < -k_h^{1/6}$*, we have for* $g \in [-\ell/K_4, -k_h^{1/6})$ *that*

$$\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{K}_h^{\mathrm{c}}) \ll e^{-c_2\ell} \min\{k_h, h\}^{-5/4} k_h^{3/2} (|g|+1)^2 e^{c_2 g} e^{-c_1 c_2 K/2} \varphi_{(\log x)^2,\delta}(\ell) e^{\frac{K_{10}(\log k_h)(\ell+g)}{k_h}}. \quad (64)$$

Remark 7. It is instructive to compare Lemma 20 with Lemma 17. First, the two results are based on different local ballot events: Lemma 17 excludes the event $\mathscr{E}_{1,\ell} \cup \mathscr{E}_{2,\ell}$, and Lemma 20 excludes the event $\mathscr{E}_{1,\ell} \cup \mathscr{E}_{3,\ell} \cup \mathscr{E}_{4,\ell}$. Second, both results require a lower bound for k_h, which greatly helps dealing with the extra term $e^{(c_2-\hat\lambda)(m_{(\log x)^2} - m_{h-K} - m_{k_h} + \ell + g)}$ in the ballot probabilities. Third, for the case $\ell > K_6 \log\log x$, we need extra preciseness in controlling the ballot probabilities, since the bound used in Lemma 17 is not tight for ℓ large. This stems from Lemmas 26 and 27 in Appendix C.2, and results in the terms involving the function $\Phi_{k_h,\delta}$ in Lemma 20. The proofs are quite similar, both applying ballot upper bounds under a proper change of measure.

Proof. The barrier given by (56) starts at location $\widetilde{x} = (\widetilde{x} - \ell) + \ell$ at time $\widetilde{t}_x$ (where we recall $\widetilde{x} = x - m_{(\log x)^2}$) and ends at location

$$\widetilde{x} + \frac{k_h}{(\log x)^2} m_{(\log x)^2} + \frac{2K_8}{c_2} (\log \min\{k_h, h\})_+ \leqslant x - m_{h-K} + g + (O(\log \min\{k_h, h\}) - g - \frac{c_1 K}{2})$$

at time $\widetilde{t}_x + k_h = t_x - h$. Note that since $g \geqslant -k_h^{1/6}$, we have $O(\log \min\{k_h, h\}) - g - c_1 K/2 \ll k_h^{1/6}$. Define $\hat\lambda := I/(m_{k_h}/k_h)$. For $g \geqslant 0$, the ballot upper bounds (Lemmas 26 and 27, together with Remark 9) under a change of measure (identically as in the proof of Proposition 10) then gives

$$\begin{aligned}
&\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{K}_h^{\mathrm{c}}) \\
&\ll \rho^{k_h} (\rho^{-k_h} k_h^{3/2} e^{-\hat\lambda(x - m_{h-K} + g - (\widetilde{x} - \ell) - m_{k_h})}) \Phi_{k_h,\delta}((\log \min\{k_h, h\} - g)_+, \ell) \\
&\ll e^{-c_2(\ell+g)} \min\{k_h, h\}^{-3/2} k_h^{3/2} e^{-c_1 c_2 K/2} \Phi_{k_h,\delta}((\log \min\{k_h, h\} - g)_+, \ell) e^{(c_2-\hat\lambda)(m_{(\log x)^2} - m_{h-K} - m_{k_h} + \ell + g)}.
\end{aligned}$$

To proceed, we need control of the final term $e^{(c_2-\hat\lambda)(m_{(\log x)^2} - m_{h-K} - m_{k_h} + \ell + g)}$. Recall from the proof of Proposition 10 that $c_2 - \hat\lambda \leqslant K_{10} (\log k_h)/k_h$ for some $K_{10} > 0$. Since $k_h \geqslant K_9 \ell$ and $\ell > K_6 \log\log x \to \infty$ as $x \to \infty$, we may assume that $c_2 - \hat\lambda \leqslant c_2/6$ by letting x be large enough. In this case,

$$e^{(c_2-\hat\lambda)(m_{(\log x)^2} - m_{h-K} - m_{k_h})} \leqslant e^{c_2(m_{(\log x)^2} - m_{h-K} - m_{k_h})/6} \ll \min\{k_h, h\}^{1/4}.$$

Combining the above leads to (62).

For the case $g < 0$, we need to exploit the rare event that two independent descendants run distances $m_{h-K} - g$ for time $h - K$ (given by Lemma 28). Applying the same arguments as in Lemma 12 and using the ballot upper bounds (Lemmas 26 and 27),

$$\begin{aligned}
&\mathbb{Q}^\ell(\mathscr{B}_h \cap \mathscr{C}_g \cap \mathscr{K}_h^{\mathrm{c}}) \\
&\ll \rho^{k_h} \big(\rho^{-k_h} k_h^{3/2} e^{-\hat\lambda(x - m_{h-K} + g - (\widetilde{x} - \ell) - m_{k_h})}\big) \Phi_{k_h,\delta}((\log \min\{k_h, h\} - g)_+, \ell) (|g|+1)^2 e^{c_2 g} \\
&\ll e^{-c_2\ell} \min\{k_h, h\}^{-3/2} k_h^{3/2} (|g|+1)^2 e^{c_2 g} e^{-c_1 c_2 K/2} \Phi_{k_h,\delta}((\log \min\{k_h, h\} - g)_+, \ell) e^{(c_2-\hat\lambda)(m_{(\log x)^2} - m_{h-K} - m_{k_h} + \ell + g)}.
\end{aligned}$$

The rest follows similarly as the case $g \geqslant 0$.

Finally, for $g \in [-\ell/K_4, -k_h^{1/6})$ we apply the same proof as above while bounding the ballot probability by $\varphi_{(\log x)^2,\delta}(\ell)$ for some $\delta > 0$, instead of $\Phi_{k_h,\delta}((\log \min\{k_h, h\} - g)_+, \ell)$. We omit the details here. ☐

Before proceeding, it is helpful to simplify the quantity $\Phi_{k_h,\delta}((\log\min\{k_h,h\}-g)_+,\ell)$ appearing in Lemma 20 a bit. By adjusting the constants δ and L and since $k_h \leqslant (\log x)^2$, it holds that

$$
\begin{aligned}
&k_h^{3/2}\Phi_{k_h,\delta}((\log\min\{k_h,h\}-g)_+,\ell)\\
&\ll \begin{cases}\ell(\log\min\{k_h,h\}-g)_+ & \text{for all } \ell > K_6\log\log x;\\ (\log\min\{k_h,h\}-g)_+(\log x)^{1/3}+\ell(\log\min\{k_h,h\}-g)_+e^{-\frac{\delta\ell^2}{(\log x)^2}} & \text{if } \ell \geqslant \frac{\log x}{L};\\ \varphi_{(\log x)^2,\delta}(\ell) & \text{for all } \ell > K_6\log\log x.\end{cases}
\end{aligned}
\tag{65}
$$

Recall also (26) and (59).

Lemma 21 (first passage contribution)**.** *For $\ell > K_6\log\log x$, there exists $C(K)>0$ such that*

$$
\begin{aligned}
&\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}}, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}_\ell^{\mathrm{c}})\\
&\ll e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\Psi_{(\log x)^2,\delta}(\ell)\big(e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}+(\log\log x)^{d+3}\ell^{-1/8}e^{\frac{\log\ell\log\log x}{8\ell}}\big)\\
&\qquad\qquad\qquad\qquad + C(K)x^{-\frac{d-1}{2}}\ell^3e^{-c_2\ell}e^{-c_2\ell/(2K_4)},
\end{aligned}
$$

where the asymptotic constant in $\ll$ does not depend on K,x,ℓ but may depend on ε.

Proof. In the following, the asymptotic constants in $\ll$ may depend on ε but not on K. We condition on $\mathscr{B}_h\cap\mathscr{C}_g$ and divide into four cases: $g\geqslant K_5\log h$, $0\leqslant g<K_5\log h$, $-\ell/K_4\leqslant g<0$, and $g<-\ell/K_4$.[12]

Case (a): $g\geqslant K_5\log h$. We apply Lemma 13 and (62) of Lemma 20 to obtain

$$
\begin{aligned}
&\sum_{h=K}^{(\log x)^2-K_9\ell}\ \sum_{g>K_5\log h} h^{d-1}x^{-\frac{d-1}{2}}\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{K}_h^{\mathrm{c}})\\
&\ll x^{-\frac{d-1}{2}}e^{-c_2\ell}e^{-c_1c_2K/2}\sum_{h=K}^{(\log x)^2-K_9\ell} h^{d-1}\min\{k_h,h\}^{-5/4}k_h^{3/2}\\
&\qquad\qquad\times\sum_{g>K_5\log h}e^{-c_2g}\Phi_{k_h,\delta}((\log\min\{k_h,h\}-g)_+,\ell)e^{\frac{K_{10}(\log k_h)(\ell+g)}{k_h}}\\
&\ll x^{-\frac{d-1}{2}}e^{-c_2\ell}e^{-c_1c_2K/2}\sum_{h=K}^{(\log x)^2-K_9\ell} h^{d-1-c_2K_5}\min\{k_h,h\}^{-5/4}k_h^{3/2}\Phi_{k_h,\delta}((\log\min\{k_h,h\})_+,\ell)e^{K_{10}\frac{(\log k_h)(\ell+K_5\log h)}{k_h}}.
\end{aligned}
$$

We split the sum over h depending on whether $k_h\leqslant h$ or $k_h>h$ (recalling the definition that $k_h=(\log x)^2-h$). We have for K_5 large enough, by (68) below,

$$
\begin{aligned}
&\sum_{h=K}^{(\log x)^2/2} h^{d-1-c_2K_5}\min\{k_h,h\}^{-5/4}k_h^{3/2}\Phi_{k_h,\delta}((\log\min\{k_h,h\})_+,\ell)e^{K_{10}\frac{(\log k_h)(\ell+K_5\log h)}{k_h}}\\
&\ll \sum_{h=K}^{(\log x)^2/2} h^{-100}k_h^{3/2}\Phi_{k_h,\delta}((\log h)_+,\ell)e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}\\
&\ll \Psi_{(\log x)^2,\delta}(\ell)e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}},
\end{aligned}
$$

[12]The three cases $g\geqslant K_5\log h$, $0\leqslant g<K_5\log h$, and $g<0$ as discussed in the proof of Lemma 18 do not suffice due to (53).

and

$$\begin{aligned}
&\sum_{h=(\log x)^2/2}^{(\log x)^2-K_9\ell} h^{d-1-c_2K_5}\min\{k_h,h\}^{-5/4}k_h^{3/2}\Phi_{k_h,\delta}((\log\min\{k_h,h\})_+,\ell)e^{K_{10}\frac{(\log k_h)(\ell+K_5\log h)}{k_h}}\\
&\ll \sum_{h=(\log x)^2/2}^{(\log x)^2-K_9\ell} h^{-100}k_h^{-5/4}k_h^{3/2}\Phi_{k_h,\delta}((\log k_h)_+,\ell)e^{K_{10}\frac{(\log k_h)(\ell+2K_5\log\log x)}{k_h}}\\
&\ll (\log x)^{-100}\sum_{h=(\log x)^2/2}^{(\log x)^2-K_9\ell} k_h^{-5/4}k_h^{3/2}\Phi_{k_h,\delta}((\log k_h)_+,\ell)\ell^{K_{10}/K_9}\\
&\ll (\log x)^{-100}\Psi_{(\log x)^2,\delta}(\ell)\ell^{K_{10}/K_9-1/4}\ll \Psi_{(\log x)^2,\delta}(\ell),
\end{aligned}$$

where we have used $k_h\geqslant K_9\ell$ and $\ell>K_6\log\log x\to\infty$ in the second step, and K_9 picked large enough (depending only on K_{10}) in the last step. Altogether, we conclude that

$$\sum_{h=K}^{(\log x)^2-K_9\ell}\sum_{g>K_5\log h} h^{d-1}x^{-\frac{d-1}{2}}\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{K}_h^{\mathrm{c}})\ll x^{-\frac{d-1}{2}}e^{-c_2\ell}e^{-c_1c_2K/2}\Psi_{(\log x)^2,\delta}(\ell)e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}.$$

In the other three cases, we will use, within the corresponding regions for g,

$$\begin{aligned}
&\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}}, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}_\ell^{\mathrm{c}})\\
&\leqslant \sum_{h=K}^{(\log x)^2}\sum_{g\in\mathbb{Z}}\sum_{\mathbf{u}\in\mathbb{Z}^{d-1}}\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}_\ell^{\mathrm{c}}\mid \mathscr{D}_{\mathbf{u}}\cap\mathscr{C}_g\cap\mathscr{B}_h\cap\mathscr{K}_h^{\mathrm{c}})\\
&\qquad\times\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{D}_{\mathbf{u}}\cap\mathscr{K}_h^{\mathrm{c}}).
\end{aligned}\tag{66}$$

For the middle two cases $-\ell/K_4\leqslant g<K_5\log h$, the asymptotic constants do not depend on the lag time $K>0$ (recall that $t_{x,K}=t_x-K$). The first conditional probability in each summand in (66) can be controlled in the same way as (48). We may also apply the same argument in Lemma 17, using now (60), to obtain bounds on $\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{D}_{\mathbf{u}}\cap\mathscr{K}_h^{\mathrm{c}})$. This amounts to multiplying the right-hand sides of (62) and (63) by $x^{-\frac{d-1}{2}}$.

Case (b): $0\leqslant g<K_5\log h$. Recall that on the event $\mathscr{E}_{3,\ell}^c$ we removed $h\in[(\log x)^2-K_9\ell,(\log x)^2]$. We further scrutinize the term $e^{K_{10}(\log k_h)(\ell+g)/k_h}$ that appears in (62). Note that K_{10} here does not depend on the other constants $K_1,\dots,K_9$. Since $k_h\geqslant K_9\ell$, $0\leqslant g<K_5\log h$, and $h\leqslant(\log x)^2$, we have for $k_h\leqslant h$,

$$e^{K_{10}\frac{(\log k_h)(\ell+g)}{k_h}}\leqslant e^{\frac{2K_{10}\log k_h\log\log x}{k_h}+\frac{K_{10}\ell\log k_h}{K_9\ell}}\leqslant e^{\frac{\log\ell\log\log x}{8\ell}}k_h^{1/8}\tag{67}$$

for K_9 picked large enough depending only on K_{10}. For $k_h\geqslant h$ (and hence $k_h\geqslant(\log x)^2/2$ and $g/k_h=O(1)$),

$$e^{K_{10}\frac{(\log k_h)(\ell+g)}{k_h}}\ll e^{K_{10}\frac{(\log k_h)\ell}{k_h}}\leqslant e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}.\tag{68}$$

For $\|\mathbf{u}\|\leqslant\sqrt{h}\log h$ and $0\leqslant g<K_5\log h$ (note that we may start the sum from $h=K$), we compute using (62)

of Lemma 20 in the first step and (67) and (68) in the third step that

$$\begin{aligned}
&\sum_{h=K}^{(\log x)^2-K_9\ell}\sum_{g=0}^{K_5\log h}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}}\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}_\ell^{\mathrm{c}}\mid \mathscr{D}_{\mathbf{u}}\cap\mathscr{C}_g\cap\mathscr{B}_h\cap\mathscr{K}_h^{\mathrm{c}})\\
&\qquad\qquad\times\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{D}_{\mathbf{u}}\cap\mathscr{K}_h^{\mathrm{c}})\\
&\ll\sum_{h=K}^{(\log x)^2-K_9\ell}\sum_{g=0}^{K_5\log h}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}}(ge^{c_2g}(\log h)^2(h-K+1)^{-\frac{d-1}{2}})\\
&\qquad\times\big(x^{-\frac{d-1}{2}}e^{-c_2\ell}\min\{k_h,h\}^{-5/4}e^{-c_2g}e^{-c_1c_2K/2}k_h^{3/2}\Phi_{k_h,\delta}((\log\min\{k_h,h\}-g)_+,\ell)e^{\frac{K_{10}(\log k_h)(\ell+g)}{k_h}}\big)\\
&\ll e^{-c_1c_2K/2}x^{-\frac{d-1}{2}}e^{-c_2\ell}\\
&\quad\times\sum_{h=K}^{(\log x)^2-K_9\ell}(\log h)^{d+3}k_h^{3/2}\Phi_{k_h,\delta}((\log\min\{k_h,h\})_+,\ell)\min\{k_h,h\}^{-5/4}\Big(\frac{h}{h-K+1}\Big)^{\frac{d-1}{2}}e^{K_{10}\frac{(\log k_h)\ell}{k_h}}\\
&\ll e^{-c_1c_2K/2}x^{-\frac{d-1}{2}}e^{-c_2\ell}\Bigg(\sum_{h=K}^{(\log x)^2/2}(\log h)^{d+3}k_h^{3/2}\Phi_{k_h,\delta}((\log h)_+,\ell)h^{-5/4}\Big(\frac{h}{h-K+1}\Big)^{\frac{d-1}{2}}e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}\\
&\qquad\qquad+\sum_{h=(\log x)^2/2}^{(\log x)^2-K_9\ell}(\log\log x)^{d+3}k_h^{3/2}\Phi_{k_h,\delta}((\log k_h)_+,\ell)k_h^{-9/8}e^{\frac{\log\ell\log\log x}{8\ell}}\Bigg)\\
&\ll e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\Psi_{(\log x)^2,\delta}(\ell)\big(e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}+(\log\log x)^{d+3}\ell^{-1/8}e^{\frac{\log\ell\log\log x}{8\ell}}\big),
\end{aligned}$$

where in the last step we used (65) and (26). The case of $\|\mathbf{u}\|>\sqrt{h}\log h$ can be dealt with in the same way as in Section 3.2.4, leading to a contribution of $x^{-\frac{d-1}{2}}\min\{e^{-c_2\ell},1\}\varphi_{(\log x)^2,\delta}(\ell)$.

Case (c): $-\ell/K_4\leqslant g<0$. Similarly as the way we derived (67) and (68), we have

$$e^{K_{10}\frac{(\log k_h)(\ell+g)}{k_h}}\leqslant e^{K_{10}\frac{(\log k_h)\ell}{k_h}}\ll\begin{cases}e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}} & \text{if } k_h\geqslant h;\\ e^{\frac{\log\ell\log\log x}{8\ell}} & \text{if } k_h<h.\end{cases}\tag{69}$$

We first consider the subcase where $\max\{-\ell/K_4,-k_h^{1/6}\}\leqslant g<0$. Again the sum over $\|\mathbf{u}\|>\sqrt{h}\log h$ can be controlled similarly as in case (c) in the proof of Lemma 18 (which only used $g<0$ but has no constraint on the range of ℓ). Recall that restricting to the event $\mathscr{E}_{3,\ell}^{\mathrm{c}}$ allows us to remove the sum over (h,g) such that $g>-\ell/K_4$ and $h>(\log x)^2-K_9\ell$. Applying (63) of Lemma 20 and (51) in the first step, that $\#\{\mathbf{u}\in\mathbb{Z}^{d-1}:\|\mathbf{u}\|\leqslant\sqrt{h}\log h\}\ll$

$h^{\frac{d-1}{2}}(\log h)^{d-1}$ in the second step, and (69) in the fourth step, we have

$$
\begin{aligned}
&\sum_{h=K}^{(\log x)^2-K_9\ell}\ \sum_{\max\{-\ell/K_4,-k_h^{1/6}\}\leqslant g<0}\ \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} \mathbb{Q}^{\ell}(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{D}_{\mathbf{u}}\cap\mathscr{K}_h^{\mathrm{c}})\\
&\qquad\qquad\qquad\times\mathbb{Q}^{\ell}(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}_\ell^{\mathrm{c}}\mid \mathscr{D}_{\mathbf{u}}\cap\mathscr{C}_g\cap\mathscr{B}_h\cap\mathscr{K}_h^{\mathrm{c}})\\
&\ll \sum_{h=K}^{(\log x)^2-K_9\ell}\ \sum_{\max\{-\ell/K_4,-k_h^{1/6}\}\leqslant g<0}\ \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} ((\log h)^2(h-K+1)^{-\frac{d-1}{2}})\\
&\quad\times\big(x^{-\frac{d-1}{2}}e^{-c_2\ell}\min\{k_h,h\}^{-5/4}(|g|+1)^2e^{c_2g}e^{-c_1c_2K/2}k_h^{3/2}\Phi_{k_h,\delta}((\log\min\{k_h,h\}-g)_+,\ell)e^{\frac{K_{10}(\log k_h)(\ell+g)}{k_h}}\big)\\
&\ll e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\sum_{h=K}^{(\log x)^2-K_9\ell}(\log h)^{d+1}\min\{k_h,h\}^{-5/4}e^{K_{10}\frac{(\log k_h)\ell}{k_h}}\\
&\qquad\qquad\times\sum_{\max\{-\ell/K_4,-k_h^{1/6}\}\leqslant g<0}(|g|+1)^2e^{c_2g}k_h^{3/2}\Phi_{k_h,\delta}((\log\min\{k_h,h\}-g)_+,\ell)\\
&\ll e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\sum_{h=K}^{(\log x)^2-K_9\ell}(\log h)^{d+1}\min\{k_h,h\}^{-5/4}e^{K_{10}\frac{(\log k_h)\ell}{k_h}}k_h^{3/2}\Phi_{k_h,\delta}((\log\min\{k_h,h\})_+,\ell)\\
&\ll e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\Big(\sum_{h=K}^{(\log x)^2/2}(\log h)^{d+1}h^{-5/4}e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}k_h^{3/2}\Phi_{k_h,\delta}((\log h)_+,\ell)\\
&\qquad\qquad+\sum_{h=(\log x)^2/2}^{(\log x)^2-K_9\ell}(\log h)^{d+1}k_h^{-5/4}e^{\frac{\log\ell\log\log x}{8\ell}}k_h^{3/2}\Phi_{k_h,\delta}((\log k_h)_+,\ell)\Big)\\
&\ll e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\Psi_{(\log x)^2,\delta}(\ell)\big(e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}+(\log\log x)^{d+3}\ell^{-1/4}e^{\frac{\log\ell\log\log x}{8\ell}}\big),
\end{aligned}
$$

where in the last step we applied (65) and (26). In the other subcase where $-\ell/K_4\leqslant g<-k_h^{1/6}$, applying (64) of Lemma 20 yields

$$
\begin{aligned}
&\sum_{h=K}^{(\log x)^2-K_9\ell}\ \sum_{-\ell/K_4\leqslant g<-k_h^{1/6}}\ \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} \mathbb{Q}^{\ell}(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}_\ell^{\mathrm{c}}\mid \mathscr{D}_{\mathbf{u}}\cap\mathscr{C}_g\cap\mathscr{B}_h\cap\mathscr{K}_h^{\mathrm{c}})\\
&\qquad\qquad\qquad\times\mathbb{Q}^{\ell}(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{D}_{\mathbf{u}}\cap\mathscr{K}_h^{\mathrm{c}})\\
&\ll e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\sum_{h=K}^{(\log x)^2-K_9\ell}(\log h)^{d+1}\min\{k_h,h\}^{-5/4}\sum_{-\ell/K_4\leqslant g<-k_h^{1/6}}(|g|+1)^2e^{c_2g}k_h^{3/2}e^{\frac{K_{10}(\log k_h)(\ell+g)}{k_h}}\\
&\ll e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\sum_{h=K}^{(\log x)^2-K_9\ell}(\log h)^{d+1}\min\{k_h,h\}^{-5/4}k_h^{-100}\varphi_{(\log x)^2,\delta}(\ell)e^{\frac{K_{10}(\log k_h)(\ell+g)}{k_h}}\\
&\ll e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\Psi_{(\log x)^2,\delta}(\ell)\big(e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}}+(\log\log x)^{d+3}\ell^{-1/4}e^{\frac{\log\ell\log\log x}{8\ell}}\big),
\end{aligned}
$$

where in the last step we also used $\varphi_{(\log x)^2,\delta}(\ell)\ll\Psi_{(\log x)^2,\delta}(\ell)$ for $\ell>K_6\log\log x$.

Case (d): $g<-\ell/K_4$. In this case, we cannot use Lemma 20, but we use Lemma 17. The sum over $\|\mathbf{u}\|>\sqrt{h}\log h$ can be controlled similarly as in case (c) in the proof of Lemma 18. We obtain using (51) and Lemma 17 that

$$
\begin{aligned}
&\sum_{h=K}^{(\log x)^2} \sum_{g<-\ell/K_4} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} \mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x \mid \mathscr{D}_{\mathbf{u}}\cap\mathscr{C}_g\cap\mathscr{B}_h\cap\mathscr{K}_h^{\mathrm{c}}) \\
&\qquad\qquad\qquad \times \mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{D}_{\mathbf{u}}\cap\mathscr{K}_h^{\mathrm{c}}) \\
&\leqslant \sum_{h=K}^{(\log x)^2} \sum_{g<-\ell/K_4} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} \mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}},\, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x \mid \mathscr{D}_{\mathbf{u}}\cap\mathscr{C}_g\cap\mathscr{B}_h)\,\mathbb{Q}^\ell(\mathscr{B}_h\cap\mathscr{C}_g\cap\mathscr{D}_{\mathbf{u}}) \\
&\ll x^{-\frac{d-1}{2}} \sum_{h=K}^{(\log x)^2} \sum_{g<-\ell/K_4} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^{d-1}\\ \|\mathbf{u}\|\leqslant\sqrt{h}\log h}} ((\log h)^2(h-K+1)^{-\frac{d-1}{2}}) \\
&\qquad\qquad \times \Big(\min\Big\{1, (|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\Big\}\min\{1, ((|g|+1)e^{c_2 g})^2\}\Big) \\
&\ll x^{-\frac{d-1}{2}} \sum_{h=K}^{(\log x)^2} (\log h)^{d+1} \sum_{g<-\ell/K_4} \min\Big\{1, (|g+\ell|+1)e^{-c_2(g+\ell)}\varphi_{(\log x)^2,\delta}(g+\ell)\Big\}(|g|+1)^2 e^{2c_2 g}\Big(\frac{h}{h-K+1}\Big)^{\frac{d-1}{2}} \\
&\ll x^{-\frac{d-1}{2}} \sum_{h=K}^{(\log x)^2} (\log h)^{d+1}\Big(\ell^2 e^{-2c_2\ell} + \ell^3 e^{-c_2\ell}e^{-c_2\ell/K_4}\Big)\Big(\frac{h}{h-K+1}\Big)^{\frac{d-1}{2}} \\
&\ll C(K)x^{-\frac{d-1}{2}}(\log x)^3 e^{-c_2\ell}e^{-c_2\ell/(2K_4)},
\end{aligned}
$$

where in the last step we bounded $(\log h)^{d+1}$ by $\log x$.

Combining the above four cases finishes the proof. □

3.2.7 Combining everything above—proof of Theorem 11

Our goal is to bound from above the quantity

$$\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}}, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x).$$

We divide into three cases according to the range of ℓ.

- Case I: $\ell < -K_3\log\log x$. We apply Lemma 13 to bound directly

$$\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}}, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x) \ll (\log x)^{2(d-1)}x^{-\frac{d-1}{2}}.$$

- Case II: $-K_3\log\log x\leqslant\ell\leqslant K_6\log\log x$. We get from Lemma 18 that while excluding the local ballot event $\mathscr{E}_\ell^*$, the rest satisfies

$$\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}}, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}_{1,\ell}^{\mathrm{c}}) \ll C(\varepsilon,K)(\log\log x)^{K_7}(\log x)x^{-\frac{d-1}{2}}e^{-c_2\ell}.$$

  Combining this with (35) and (39) yields Theorem 11 for $-K_3\log\log x\leqslant\ell\leqslant K_6\log\log x$.

- Case III: $\ell > K_6\log\log x$. The event $\mathscr{E}_{3,\ell}$ contributes

$$e^{-(c_2+\delta/4)\ell}(\log x)^{2(d-1)}x^{-\frac{d-1}{2}} \tag{70}$$

  by Lemma 19. We get from Lemma 21 that while excluding the ballot event $\mathscr{E}_\ell$, the rest satisfies

$$
\begin{aligned}
&\mathbb{Q}^\ell(\exists\, w\in V_{t_{x,K}}, w\succ v,\, \boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x,\, \mathscr{E}_\ell^{\mathrm{c}}) \\
&\ll C(\varepsilon)e^{-c_1c_2K/4}x^{-\frac{d-1}{2}}e^{-c_2\ell}\Psi_{(\log x)^2,\delta}(\ell)\big(e^{2K_{10}\frac{\ell\log\log x}{(\log x)^2}} + (\log\log x)^{d+3}\ell^{-1/8}e^{\frac{\log\ell\log\log x}{8\ell}}\big) \\
&\qquad\qquad + C(K)x^{-\frac{d-1}{2}}(\log x)^3 e^{-c_2\ell}e^{-c_2\ell/(2K_4)},
\end{aligned}
$$

  The second term $C(K)x^{-\frac{d-1}{2}}(\log x)^3e^{-c_2\ell}e^{-c_2\ell/(2K_4)}$ on the right-hand side above together with (70) contribute at most $C(\varepsilon,K)e^{-(c_2+\delta/4)\ell}(\log x)^{3d}x^{-\frac{d-1}{2}}$ for some $\delta>0$. Taking into account (35), Lemma 19, and (58) leads to the desired bound in Theorem 11 for $\ell > K_6\log\log x$.

These considerations conclude the proof, given the definition of $I_{\ell,x}$ above Theorem 11.

4 The general non-spherically symmetric case

Our cluster approach in the spherically symmetric case (Theorem 1) also extends to the general case (Theorem 2). Unfortunately, the notation becomes much heavier, although it is intuitively clear how the proof can be modified, as we will explain below. We have chosen to focus on the spherically symmetric case for clarity of our presentation and to avoid repetitions. In this section, we comment on the necessary changes to prove Theorem 2, and the details are left to the interested reader.

Overview. Essentially, our approach reduces the study of the d-dimensional BRW to that of its projection onto a certain one-dimensional direction (pivot) $\mathbf{c}_2 \in \mathbb{R}^d$ such that the first passage event to B_x is reasonably close to a certain first passage event of the *projected* BRW (to the projection of the target B_x). The underlying mechanism of this approximation is that the range of a non-symmetric BRW grows roughly as a convex shape that linearly expands in time, and $\mathbf{c}_2$ is the normal vector to the tangent hyperplane between the convex shape and the ball B_x. For example, in the spherically symmetric case, $\mathbf{c}_2$ is along the direction of the first coordinate (i.e., a constant multiple of $\mathbf{e}_1$). The non-symmetric case requires the same techniques, up to finding the correct pivot $\mathbf{c}_2$ based on the rate function $\widehat{I}(\boldsymbol{\xi})$, which is given below (3).

Let us formulate the new projection. In the spherically symmetric case, we follow the projection

$$\boldsymbol{\eta}_{v,n}(k) = \eta_{v,n}(k)\,\mathbf{e}_1 + (0, \widehat{\boldsymbol{\eta}}_{v,n}(k)).$$

Instead, we now decompose

$$\boldsymbol{\eta}_{v,n}(k) = \eta^{\mathbf{c}_2}_{v,n}(k)\,\mathbf{c}_2 + \widehat{\boldsymbol{\eta}}^{\mathbf{c}_2}_{v,n}(k),$$

where $\widehat{\boldsymbol{\eta}}^{\mathbf{c}_2}_{v,n}(k) \in \mathbb{R}^d$ is perpendicular to $\mathbf{c}_2$, i.e., $\widehat{\boldsymbol{\eta}}^{\mathbf{c}_2}_{v,n}(k) \cdot \mathbf{c}_2 = 0$. The proof of Theorem 2 is mostly verbatim, while in the two paragraphs below we spell out a few details that differ from the proof of Theorem 1.

Identifying the constants in the preliminary results. Let us re-discover the formula (4) based on a calculation using the BRW projected onto $\mathbf{c}_2$. The one-step jump distribution is $\xi^{\mathbf{c}_2} := \boldsymbol{\xi} \cdot \mathbf{c}_2$. Using the definition $\widehat{I}(\widehat{c}_1\mathbf{e}_1) = \log\rho$ and $\mathbf{c}_2 = \nabla\widehat{I}(\widehat{c}_1\mathbf{e}_1)$, we have

$$\sup_{\boldsymbol{\lambda}\in\mathbb{R}^d}\Big(\widehat{c}_1\boldsymbol{\lambda}\cdot\mathbf{e}_1 - \log\mathbb{E}[e^{\boldsymbol{\lambda}\cdot\boldsymbol{\xi}}]\Big) = \log\rho \quad \text{and} \quad \widehat{c}_1\mathbf{e}_1 = \frac{\mathbb{E}[\boldsymbol{\xi}e^{\mathbf{c}_2\cdot\boldsymbol{\xi}}]}{\mathbb{E}[e^{\mathbf{c}_2\cdot\boldsymbol{\xi}}]}.$$

It is then straightforward to check that the supremum in

$$I^{\xi^{\mathbf{c}_2}}(\widehat{c}_1\mathbf{e}_1\cdot\mathbf{c}_2) = \sup_{\lambda\in\mathbb{R}}\Big(\lambda\widehat{c}_1\mathbf{e}_1\cdot\mathbf{c}_2 - \log\mathbb{E}[e^{\lambda\xi^{\mathbf{c}_2}}]\Big)$$

is attained at $\widetilde{\lambda} = 1$ and the value of the supremum is $\log\rho$. This has two consequences. First, the linear speed of the BRW with jump $\xi^{\mathbf{c}_2}$ is $\widehat{c}_1\mathbf{e}_1\cdot\mathbf{c}_2$, which gives the linear coefficient in (4). Second, when dealing with the projected BRW, the analogue of the constant c_2 in the results presented in Section 2.1 becomes $\widetilde{\lambda} = 1$. Consequently, the logarithm correction term is

$$\frac{d+2}{2\widetilde{\lambda}\widehat{c}_1\mathbf{e}_1\cdot\mathbf{c}_2}\log x = \frac{d+2}{2\widehat{c}_1\mathbf{e}_1\cdot\mathbf{c}_2}\log x,$$

giving the logarithmic correction term in (4).

Conditional local CLT in the direction $\mathbf{c}_2$. While the cluster structure remains unchanged for the one-dimensional BRW projected onto $\mathbf{c}_2$, certain modification is required to turn the size of the clusters to the local hitting probabilities (that is, given a trajectory that advances in the direction $\mathbf{c}_2$, we compute the chance that it reaches the ball B_x). In the spherically symmetric case, this is driven by the conditional local CLT (Lemmas 9 and 14). The selection of the vector $\mathbf{c}_2$ is exactly such that the analogous local CLT holds in the new direction $\mathbf{c}_2$. We showcase this by providing the proof to a more general version of Lemma 9, given by Lemma 30 below. The same extension to Lemma 14 can be done similarly.

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References

[1] Louigi Addario-Berry and Bruce Reed. Minima in branching random walks. *Annals of Probability*, 37(3):1044–1079, 2009.

[2] Elie Aïdékon. Convergence in law of the minimum of a branching random walk. *Annals of Probability*, 41(3A):1362–1426, 2013.

[3] Louis-Pierre Arguin. Extrema of log-correlated random variables. *Advances in Disordered Systems, Random Processes and Some Applications*, page 166, 2016.

[4] Louis-Pierre Arguin, Anton Bovier, and Nicola Kistler. Genealogy of extremal particles of branching brownian motion. *Communications on Pure and Applied Mathematics*, 64(12):1647–1676, 2011.

[5] Krishna B Athreya and Peter E Ney. *Branching Processes*. Courier Corporation, 2004.

[6] Julien Berestycki, Yujin H Kim, Eyal Lubetzky, Bastien Mallein, and Ofer Zeitouni. The extremal point process of branching brownian motion in $\mathbb{R}^d$. *The Annals of Probability*, 52(3):955–982, 2024.

[7] Viktor Bezborodov and Nina Gantert. The maximal displacement of radially symmetric branching random walk in $\mathbb{R}^d$. *arXiv preprint arXiv:2309.14738*, 2023.

[8] Jose Blanchet, Wei Cai, Shaswat Mohanty, and Zhenyuan Zhang. On the first passage times of branching random walks in $\mathbb{R}^d$. *arXiv preprint arXiv:2404.09064*, 2024.

[9] Maury Bramson, Jian Ding, and Ofer Zeitouni. Convergence in law of the maximum of nonlattice branching random walk. *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, 52(4):1897–1924, 2016.

[10] Maury Bramson and Ofer Zeitouni. Tightness for a family of recursion equations. *Annals of Probability*, 37(2):615–653, 2009.

[11] Dariusz Buraczewski and Mariusz Maślanka. Large deviation estimates for branching random walks. *ESAIM: Probability and Statistics*, 23:823–840, 2019.

[12] Aser Cortines, Lisa Hartung, and Oren Louidor. The structure of extreme level sets in branching brownian motion. *The Annals of Probability*, 47(4):2257–2302, 2019.

[13] Aser Cortines, Lisa Hartung, and Oren Louidor. More on the structure of extreme level sets in branching brownian motion. *Electronic Communications in Probability*, 26(2):1–14, 2021.

[14] Amir Dembo and Ofer Zeitouni. *Large Deviations Techniques and Applications*. Springer, 1998.

[15] Denis Denisov and Vitali Wachtel. Random walks in cones. *Annals of Probability*, 43(3):992–1044, 2015.

[16] Arnaud Ducrot. On the large time behaviour of the multi-dimensional Fisher–KPP equation with compactly supported initial data. *Nonlinearity*, 28(4):1043, 2015.

[17] Ronald Aylmer Fisher. The wave of advance of advantageous genes. *Annals of Eugenics*, 7(4):355–369, 1937.

[18] Nina Gantert and Thomas Höfelsauer. Large deviations for the maximum of a branching random walk. *Electronic Communications in Probability*, 23(34):1–12, 2018.

[19] Jürgen Gärtner. Location of wave fronts for the multi-dimensional K-P-P equation and Brownian first exit densities. *Mathematische Nachrichten*, 105(1):317–351, 1982.

[20] Ion Grama and Hui Xiao. Conditioned local limit theorems for random walks on the real line. *arXiv preprint arXiv:2110.05123, to appear in Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, 2024.

[21] Lisa Hartung, Oren Louidor, and Tianqi Wu. On the growth of the extremal and cluster level sets in branching brownian motion. *arXiv preprint arXiv:2405.17634*, 2024.

[22] Yueyun Hu. How big is the minimum of a branching random walk? *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, 52:233–260, 2016.

[23] Yueyun Hu and Zhan Shi. Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Annals of Probability*, 37(2):742–789, 2009.

[24] Predrag R Jelenković and Mariana Olvera-Cravioto. Maximums on trees. *Stochastic Processes and their Applications*, 125(1):217–232, 2015.

[25] Ioannis Karatzas and Steven Shreve. *Brownian Motion and Stochastic Calculus*, volume 113. Springer, 2014.

[26] Yujin H Kim, Eyal Lubetzky, and Ofer Zeitouni. The maximum of branching Brownian motion in $\mathbb{R}^d$. *The Annals of Applied Probability*, 33(2):1515–1568, 2023.

[27] Yujin H Kim and Ofer Zeitouni. The shape of the front of multidimensional branching brownian motion. *arXiv preprint arXiv:2401.12431*, 2024.

[28] Andrei Kolmogorov, Ivan Petrovskii, and Nikolai Piskunov. Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologigue. *Moscow Univ. Bull. Ser. Internat. Sect. A*, 1:1, 1937.

[29] Wolfgang König. Branching random walks in random environment: a survey. In *Probabilistic Structures in Evolution*, pages 23–41. EMS Press, Berlin, 2021.

[30] Mark Kot, Jan Medlock, Timothy Reluga, and D Brian Walton. Stochasticity, invasions, and branching random walks. *Theoretical Population Biology*, 66(3):175–184, 2004.

[31] Lianghui Luo. Precise upper deviation estimates for the maximum of a branching random walk. *arXiv preprint arXiv:2403.03687*, 2024.

[32] Thomas Madaule. Convergence in law for the branching random walk seen from its tip. *Journal of Theoretical Probability*, 30:27–63, 2017.

[33] Bastien Mallein. Maximal displacement in the d-dimensional branching Brownian motion. *Electronic Communications in Probability*, 20:1–12, 2015.

[34] Bastien Mallein. Asymptotic of the maximal displacement in a branching random walk. *Graduate J. Math*, 1(2):92–104, 2016.

[35] Mehmet Öz. Large deviations for local mass of branching brownian motion. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 17:711–731, 2020.

[36] Mehmet Öz. On the density of branching Brownian motion. *Hacettepe Journal of Mathematics & Statistics*, 52(1):229–247, 2023.

[37] Valentin Vladimirovich Petrov. On the probabilities of large deviations for sums of independent random variables. *Theory of Probability & Its Applications*, 10(2):287–298, 1965.

[38] R Tyrrell Rockafellar. *Convex Analysis*, volume 18. Princeton University Press, 1970.

[39] Jean-Michel Roquejoffre, Luca Rossi, and Violaine Roussier-Michon. Sharp large time behaviour in N-dimensional Fisher-KPP equations. *Discrete and Continuous Dynamical Systems A*, 39:7265–7290, 2019.

[40] Zhan Shi. *Branching Random Walks.* Volume 2151 of Lecture Notes in Mathematics. Springer, 2015.

[41] Michel Talagrand. *Upper and Lower Bounds for Stochastic Processes: Decomposition Theorems*, volume 60. Springer Nature, 2022.

[42] Kohei Uchiyama. Spatial growth of a branching process of particles living in $\mathbb{R}^d$. *The Annals of Probability*, 10(4):896–918, 1982.

[43] Ofer Zeitouni. Branching random walks and Gaussian fields. *Probability and Statistical Physics in St. Petersburg*, 91:437–471, 2016.

[44] Shuxiong Zhang. Lower deviation probabilities for level sets of the branching random walk. *Journal of Theoretical Probability*, 36(2):811–844, 2023.

[45] Shuxiong Zhang. Large deviation probabilities for the range of a d-dimensional supercritical branching random walk. *Applied Mathematics and Computation*, 462:128344, 2024.

[46] Zhenyuan Zhang, Shaswat Mohanty, Jose Blanchet, and Wei Cai. Modeling shortest paths in polymeric networks using spatial branching processes. *Journal of the Mechanics and Physics of Solids*, page 105636, 2024.

A Index of frequently used notation

	Deterministic quantities
d	Underlying dimension of the BRW, $d \geqslant 1$
$\{p_j\}_{j\geqslant 0}$	Reproduction law of the BRW
$\boldsymbol{\xi}$	Jump distribution of the BRW
$I(x)$	Large deviation rate function for the first coordinate ξ of $\boldsymbol{\xi}$
$\widehat{I}(\mathbf{x})$	Large deviation rate function for $\boldsymbol{\xi}$
ρ	Expected number of descendants at time one, $\rho = \sum_{j\geqslant 1} jp_j$
c_1	Defined through $I(c_1) = \log\rho$
c_2	$I'(c_1)$
$\widehat{c}_1$	Defined through $\widehat{I}(\widehat{c}_1, \mathbf{0}) = \log\rho$
$\mathbf{c}_2$	$\nabla\widehat{I}(\widehat{c}_1, \mathbf{0})$
m_n	(One-dimensional) maximum asymptotic $c_1 n - \frac{3}{2c_2}\log n$
t_x	First passage time asymptotic $\frac{x}{c_1} + \frac{d+2}{2c_2c_1}\log x$ in dimension d
$t_{x,K}$	$t_x - K$
$\widetilde{x}$	$x - m_{(\log x)^2}$
$\widetilde{t}_x$	$t_x - (\log x)^2$ (with the exception of Appendix D)
k_h	$(\log x)^2 - h$
$\varphi_{n,\delta}(i)$	$e^{-\delta\lvert i\rvert\min(\frac{\lvert i\rvert}{n},1)}$
$\Phi_{n,\delta}(x,y)$	Defined by (61)
$\Psi_{(\log x)^2,\delta}(\ell)$	Defined by (26)
	Events
$\mathscr{B}_h$	$h = \max\{\widetilde{h} : \exists\, v_1 \in V_{t_x-\widetilde{h}}, v_2, v_3 \in V_{t_{x,K}}, v_2, v_3 \succ v_1 \succ v, \eta_{v_i,t_{x,K}}(t_{x,K}) \geqslant x, i = 2,3\}$
v_{lca}	The particle $v_1 \in V_{t_x-h}$ realizing the maximum above (latest common ancestor)
$\mathscr{C}_g$	$\{\eta_{v_{\mathrm{lca}},t_x-h}(t_x-h) \in [x+g-m_{h-K}, x+g-m_{h-K}+1]\}$
$\mathscr{D}_{\mathbf{u}}$	The event that the last $d-1$ coordinates of $\boldsymbol{\eta}_{v_{\mathrm{lca}},t_x-h}(t_x-h)$ belongs to $R_{\mathbf{u}}$
$W_{\ell,h,g}$	$\{w \in V_{t_x-h} : w \succ v,\ \lVert\widehat{\boldsymbol{\eta}}_{w,t_x-h}(t_x-h)\rVert \ll h,\ \mathscr{B}_h \cap \mathscr{C}_g\}$
$\mathscr{E}_{1,\ell}$	$\bigcup_{K\leqslant h\leqslant(\log x)^2}\bigcup_{g\in\mathbb{Z}}\bigcup_{\substack{w\in V_{t_x-h}\\ w\in W_{\ell,h,g}}}\bigcup_{\substack{w'\in V_{t_{x,K}}\\ w'\succ w}}\bigcup_{t_x-h\leqslant k\leqslant t_{x,K}}\{\eta_{w',t_{x,K}}(k) > \widehat{\psi}_{g,h}(k)\}$, $\ell \geqslant -K_3\log\log x$
$\mathscr{E}_{2,\ell}$	$\bigcup_{K\leqslant h\leqslant(\log x)^2}\Big(\mathscr{B}_h \cap \Big(\bigcup_{u\in V_{t_x-h}}\Big(\bigcup_{\widetilde{t}_x\leqslant k\leqslant t_x-h}\{\eta_{u,t_x-h}(k) > \psi^*_{x,K}(k)\}\Big) \cap\Big(\bigcup_{\substack{w\in V_{t_{x,K}}\\ w\succ u}}\{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K}) \in B_x\}\Big)\Big)\Big)\Big)$, $-K_3\log\log x \leqslant \ell \leqslant K_6\log\log x$
$\mathscr{E}_{3,\ell}$	$\big(\bigcup_{g\geqslant-\ell/K_4}\mathscr{C}_g\big) \cap \big(\bigcup_{(\log x)^2-K_9\ell\leqslant h\leqslant(\log x)^2}\mathscr{B}_h\big)$, $\ell > K_6\log\log x$
$\mathscr{E}_{4,\ell}$	$\bigcup_{K\leqslant h\leqslant(\log x)^2-K_9\ell}\Big(\mathscr{B}_h \cap \Big(\bigcup_{u\in V_{t_x-h}}\Big(\bigcup_{\widetilde{t}_x\leqslant k\leqslant t_x-h}\{\eta_{u,t_x-h}(k) > \psi_{x,K}(k)\}\Big)$

$$\cap\Big(\bigcup_{\substack{w\in V_{t_{x,K}}\\ w\succ u}}\{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x\}\Big)\Big)\Big),\ \ell>K_6\log\log x$$

$\mathscr{E}_\ell^*$	$\mathscr{E}_{1,\ell}\cup\mathscr{E}_{2,\ell}$
$\mathscr{E}_\ell$	$\mathscr{E}_{1,\ell}\cup\mathscr{E}_{3,\ell}\cup\mathscr{E}_{4,\ell}$
$\mathscr{F}_{h,g}$	$\bigcup_{\substack{w\in V_{t_{x,K}}\\ w\succ v_{\rm lca}}}\bigcup_{0\leqslant k\leqslant h-K}\Big\{\eta_{w,t_{x,K}}(\widetilde{t}_x-h+k)-\eta_{w,t_{x,K}}(\widetilde{t}_x-h)$ $\geqslant L\log h-g+\frac{k}{h-K}m_{h-K}+\frac{4}{c_2}(\log\min\{k,h-K-k\})_+\Big\}$
$\mathscr{G}_{n,\beta}$	$\bigcup_{v\in V_n}\bigcup_{0\leqslant k\leqslant n}\Big\{\eta_{v,n}(k)\geqslant\frac{km_n}{n}+\beta+\frac{6}{c_2}(\log\min\{k,n-k\})_+\Big\}$
$\mathscr{H}_{\mathbf{u}}$	$\{\boldsymbol{\eta}_{v_{\rm lca},t_x-h}(t_x-h)-\boldsymbol{\eta}_{v_{\rm lca},t_x-h}(\widetilde{t}_x)\in\mathbb{R}\times R_{\mathbf{u}}\}$
$\mathscr{I}_{n,g}$	$\{\exists\, v,w\in V_n,\ {\rm lca}(v,w)=\emptyset,\ \eta_{v,n}(n)\geqslant m_n-g,\ \eta_{w,n}(n)\geqslant m_n-g\}$
$\mathscr{J}_{h,g}$	$\{\exists\, w\in V_{t_x-h},\ w\succ v,\ \eta_{w,t_x-h}(t_x-h)\in[x-m_{h-K}+g,x-m_{h-K}+g+1)\}$
$\mathscr{K}_h$	$\mathscr{B}_h\cap\Big(\bigcup_{u\in V_{t_x-h}}\Big(\bigcup_{\widetilde{t}_x\leqslant k\leqslant t_x-h}\{\eta_{u,t_x-h}(k)>\psi_{x,K}(k)\}\Big)\cap\Big(\bigcup_{\substack{w\in V_{t_{x,K}}\\ w\succ u}}\{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x\}\Big)\Big)$
$\mathscr{K}_h^*$	$\mathscr{B}_h\cap\Big(\bigcup_{u\in V_{t_x-h}}\Big(\bigcup_{\widetilde{t}_x\leqslant k\leqslant t_x-h}\{\eta_{u,t_x-h}(k)>\psi^*_{x,K}(k)\}\Big)\cap\Big(\bigcup_{\substack{w\in V_{t_{x,K}}\\ w\succ u}}\{\boldsymbol{\eta}_{w,t_{x,K}}(t_{x,K})\in B_x\}\Big)\Big)$
S	All-time survival event of the BRW

Other definitions

$\#A$	Cardinality of a finite set A
$w\succ v$	Particle w is a descendant of v
$\emptyset$	The unique particle at time zero
B_x	Unit ball centered at $\mathbf{x}=(x,0,\dots,0)\in\mathbb{R}^d$
$B_{\mathbf{z}}$	Unit ball centered at $\mathbf{z}\in\mathbb{R}^d$
$\mathbb{H}_x$	$[x,\infty)\times\mathbb{R}^{d-1}$
τ_x	First passage time of d-dimensional BRW to B_x
M_n	Maximum of one-dimensional BRW at time n
P_n	Production number, defined as $\#\{v\in V_n:\exists\, w\in V_{t_x},\ w\succ v,\ \eta_{w,t_x}(t_x)\geqslant x\}$
$R_{\mathbf{u}}$	The rectangle $[u_1,u_1+1)\times\cdots\times[u_{d-1},u_{d-1}+1)$ for $\mathbf{u}=(u_1,\dots,u_{d-1})$
V_n	The collection of particles at time step n
$\boldsymbol{\eta}_{v,n}(k)$	Location of the d-dimensional random walk that leads to $v\in V_n$ evaluated at time k
$\eta_{v,n}(k)$	The first coordinate of $\boldsymbol{\eta}_{v,n}(k)$
$\widehat{\boldsymbol{\eta}}_{v,n}(k)$	The last $d-1$ coordinates of $\boldsymbol{\eta}_{v,n}(k)$
$\mathbb{Q}^\ell=\mathbb{Q}^{\ell,v}$	The probability measure on the BRW restricted to descendants of v, conditioned on $\eta_{v,\widetilde{t}_x}(\widetilde{t}_x)\in[\widetilde{x}-\ell-1,\widetilde{x}-\ell)$ and $\mathscr{G}^{\rm c}_{t_x,K_2}$

B Escape probability of BRW

The goal of this appendix is to establish the following.

Lemma 22. *Assume (A1)–(A4). There exists $K_1>0$ such that*

$$\mathbb{P}(\|\boldsymbol{\eta}_{v,n}(n)\|\geqslant 1 \text{ for all } v\in V_n)\leqslant K_1e^{-\sqrt{n}/K_1}.$$

Remark 8. The closest result in this direction is perhaps [44], which studied convergence rates of

$$\mathbb{P}(\#\{v\in V_n:\eta_{v,n}(n)>\theta c_1n\}>e^{an})$$

for $a\in[0,\log\rho-I(\theta c_1))$. For branching Brownian motion, [35, 36] studied large deviation probabilities of the number of particles in a linearly moving ball. In particular, the analogue of Lemma 22 for BBM was established as a special case of Theorem 2.1 of [36]. Our result is quite crude (for instance, we believe that the escape probability can be improved to $O(e^{-n/L})$ based on analogues in [35]), but it suffices for our purpose. Additionally, the arguments required are relatively simple compared to the literature above.

Proof of Lemma 22. By a union bound, it suffices to work in the one-dimensional setting. The strategy is to evolve particles independently in the periods $[0,\sqrt{n}]$ and $[\sqrt{n},n]$. We show that at time $\sqrt{n}$, with high probability there are $e^{\delta\sqrt{n}}$ particles present and located in $O(\sqrt{n})$, and with high probability, a certain portion of the particles located in $O(\sqrt{n})$ at time $\sqrt{n}$ will have a descendant in $[-1,1]$ at time n.

To carry out the above plan, let $\delta_1 > 0$ be a small constant and we define the events

$$E_1 := \{\#V_{\sqrt{n}} \geqslant e^{\delta_1\sqrt{n}}\} \quad \text{and} \quad E_2 := \Big\{|M_{\sqrt{n}}| < \frac{\sqrt{n}}{\delta_1}\Big\}.$$

It follows from the main result of [44] and Theorem 3.2 of [18] that $\mathbb{P}(E_1 \cap E_2) > 1 - O(e^{-\delta_2\sqrt{n}})$ for some $\delta_2 > 0$. On the event $E_1 \cap E_2$, we may identify particles $v_j,\ 1 \leqslant j \leqslant e^{\delta_1\sqrt{n}}$, where $\eta_{v_j,\sqrt{n}}(\sqrt{n}) \in [-\sqrt{n}/\delta_1, \sqrt{n}/\delta_1]$. Let S_j denote the survival event of the particle v_j. It follows from local CLT applied to ξ (see e.g. Lemma 23 of [8]) that for some $\delta_3 > 0$,

$$\mathbb{P}\Big(\exists\, w \in V_n,\ w \succ v_j,\ |\eta_{w,n}(n)| \leqslant 1 \mid S_j\Big) \geqslant \frac{\delta_3}{\sqrt{n}},$$

and hence using independence, the event

$$E_3 := \Big\{\#\{w \in V_{n/2} : |\eta_{w,n}(n)| \leqslant 1\} > 0\Big\}$$

satisfies

$$\mathbb{P}(E_3 \mid E_1 \cap E_2) \geqslant 1 - (1 - \frac{\delta_3}{\sqrt{n}})^{e^{\delta_1\sqrt{n}}} > 1 - O(e^{-n}).$$

We thus conclude that

$$\mathbb{P}(\|\boldsymbol{\eta}_{v,n}(n)\| \geqslant 1 \text{ for all } v \in V_n) \leqslant \mathbb{P}(E_1^{\mathrm{c}} \cup E_2^{\mathrm{c}}) + \mathbb{P}(E_3^{\mathrm{c}} \mid E_1 \cap E_2) \leqslant O(e^{-\delta_2\sqrt{n}}).$$

This proves Lemma 22. □

C Some upper bounds of (conditional) ballot probabilities

C.1 A multi-dimensional ballot upper bound

The results in this appendix are essential for establishing Lemma 13 through Lemma 14. Following the seminal work of [15] on random walks in cones, we prove a multi-dimensional ballot upper bound where the random walk reaches a target in $\mathbb{R}^d$ and the path projected onto the first dimension is constrained by a linear barrier tilted by a logarithmic term. The connection to random walks in cones (i.e. collections of rays from $\mathbf{0} \in \mathbb{R}^d$ going through a certain open subset of the sphere $\mathbb{S}^{d-1}$) is realized by letting the cone be the half-space $\{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 > 0\}$. The following statements are self-contained, but we refer to Section 2.4.1 of [8] for a brief introduction to random walks in cones.

In the following, let $\{\boldsymbol{\xi}_i\}_{i \geqslant 1}$ be an i.i.d. sequence in $\mathbb{R}^d$ satisfying (A2)–(A4) and $\mathbf{S}_n = \boldsymbol{\xi}_1 + \cdots + \boldsymbol{\xi}_n$ be its partial sum. Denote by ξ_i and S_n their first coordinates. Let

$$\overline{\psi}(k) = -y - L \log \min\{k, n-k\}_+,\ \ 0 \leqslant k \leqslant n,$$

where $L > 0$ is a fixed constant. Recall that $R_{\mathbf{u}} = [u_1, u_1+1) \times \cdots \times [u_{d-1}, u_{d-1}+1)$ for $\mathbf{u} = (u_1, \ldots, u_{d-1}) \in \mathbb{R}^{d-1}$. The following result improves upon Lemma 16 of [8].

Lemma 23. *Consider $n \geqslant 1$, $1 \leqslant y \ll \log n$, and $y + a > 0$. Then*

$$\mathbb{P}(\mathbf{S}_n \in [a, a+1] \times R_{\mathbf{u}},\ S_k \geqslant \overline{\psi}(k) \text{ for all } 0 < k < n) \ll y(y+a)n^{-\frac{d+2}{2}}.$$

Lemma 24. *Consider $n \geqslant 1$, $1 \leqslant y \ll \log n$, and $y + a > 0$. It holds*

$$\mathbb{P}(\mathbf{S}_n \in [a, a+1] \times R_{\mathbf{u}},\ S_k \geqslant -y \text{ for all } 0 < k < n) \ll y(y+a)n^{-\frac{d+2}{2}}.$$

Proof. This follows from the derivation of Lemmas 27 and 28 in [15], along with their Theorem 1. Note that $p = 1$ therein since we take the cone to be the half-space $\{\mathbf{x} \in \mathbb{R}^d : x_1 > 0\}$. The results in [15] were stated in the lattice case but the derivation of Lemmas 27 and 28 depends only on the non-singularity of the jumps (which is in turn a consequence of (A5)) but not the lattice property. □

The proof of Lemma 23 follows a similar route as the proof of (11) in [9], while in the proof we apply now Lemma 24 instead of (6) of Lemma 2.1 therein.

Proof of Lemma 23. Let τ be the first time in $[0,n]$ at which S_k takes its minimum, and suppose that $a \geqslant -1/2$. We split into cases depending on the value of τ and apply a union bound. By symmetry of the function $\overline{\psi}$, we may assume $\tau < n/2$. Define $I_y := \mathbb{Z} \cap (y^7, n/2]$ and

$$\begin{aligned} \Omega_{n,y} &= \big\{\mathbf{S}_n \in [a,a+1] \times R_{\mathbf{u}},\, S_k \geqslant -y \text{ for all } 0<k<n\big\}, \\ \Omega'_{n,y} &= \big\{\mathbf{S}_n \in [a,a+1] \times R_{\mathbf{u}},\, S_k \geqslant \overline{\psi}(k) \text{ for all } 0<k<n\big\}. \end{aligned}$$

It follows from Lemma 24 that (using $a \geqslant -1/2$)

$$\begin{aligned} \mathbb{P}(\tau \in I_y; \Omega'_{n,y}) &\ll \sum_{k \in I_y} \sum_{j=0}^{y+L\log k} \frac{(j+1)(j+a)}{k^{3/2}(n-k)^{\frac{d+2}{2}}} \\ &\ll \sum_{k \in I_y} \frac{(y+L\log k+1)^2(y+L\log k+a)}{k^{3/2}(n-k)^{\frac{d+2}{2}}} \ll (y+a)y^{-3/2}n^{-\frac{d+2}{2}}. \end{aligned}$$

Next, we consider $k \leqslant y^{19/10}$. On the event $\Omega'_{n,y} \setminus \Omega_{n,y}$, we have $S_k \in [-y-L\log k, -y]$. Therefore, by independence before and after time k and Lemma 24,

$$\begin{aligned} &\mathbb{P}(\tau = k; \Omega'_{n,y} \setminus \Omega_{n,y}) \\ &\leqslant \mathbb{P}(S_k \in [-y-L\log k, -y]) \\ &\quad \times \max_{x \in [-y-L\log k, -y]} \sup_{\substack{\mathbf{u}' \in \mathbb{R}^{d-1} \\ \|\mathbf{u}'\| \geqslant \|\mathbf{u}\|/2}} \mathbb{P}(S_j \geqslant 0 \text{ for all } 1 \leqslant j \leqslant n-k;\, \mathbf{S}_{n-k} \in [a+x, a+x+1] \times R_{\mathbf{u}'}) \\ &\ll y^{-2} \times (a+y+\log k)n^{-\frac{d+2}{2}} \ll y^{-2}(y+a)n^{-\frac{d+2}{2}}, \end{aligned}$$

where the upper bound for $\mathbb{P}(S_k \in [-y-L\log k, -y])$ follows from the same reasoning as below (91) of [9].

For $k \in (y^{19/10}, y^7)$, we again apply independence of the random walk before and after time k. We have in this case the boundary of $\Omega'_{n,y}$ is at most $\ll y^{10}$ below that for $\Omega_{n,y}$ uniformly in $k \in (y^{19/10}, y^7)$. Therefore, applying Lemma 24 twice yields

$$\mathbb{P}(\tau = k; \Omega'_{n,y} \setminus \Omega_{n,y}) \ll y^{1/10}k^{-3/2} \times (y+\log k)(y+\log k+a)n^{-\frac{d+2}{2}} \ll y^{11/10}(y+a)k^{-3/2}n^{-\frac{d+2}{2}}.$$

Combining the above estimates yields that for all $\mathbf{u} \in \mathbb{R}^{d-1}$,

$$\begin{aligned} \mathbb{P}(\Omega'_{n,y} \setminus \Omega_{n,y}) &\ll (y+a)y^{-3/2}n^{-\frac{d+2}{2}} + \sum_{k=1}^{y^{19/10}} y^{-2}(y+a)n^{-\frac{d+2}{2}} + \sum_{k=y^{19/10}}^{y^7} y^{11/10}(y+a)k^{-3/2}n^{-\frac{d+2}{2}} \\ &\ll y(y+a)n^{-\frac{d+2}{2}} \end{aligned}$$

for some $\delta > 0$. Also note that Lemma 24 yields $\mathbb{P}(\Omega_{n,y}) \ll y(y+a)n^{-\frac{d+2}{2}}$. This finishes the proof for $a \geqslant -1/2$. The case $a < -1/2$ follows by reversing the random walk, using precisely the same argument at the end of the proof of (11) in [9]. ☐

C.2 A ballot upper bound involving moderate deviation

In this appendix, we revisit the recent work [20] and extract one-dimensional ballot upper bounds involving moderate deviation. Consider a one-dimensional non-lattice random walk $\{S_n\}_{n \geqslant 1}$ with centered i.i.d. jumps and finite moments of any order. The quantity of interest is

$$\mathbb{P}(x + S_n \in [y, y+1],\, x + S_n \geqslant 0 \text{ for all } 1 \leqslant k \leqslant n), \tag{71}$$

where $|x-y| \gg \sqrt{n}$. Applying the classical ballot theorem leads only to an upper bound of $\ll xyn^{-3/2}$, which does not account for the fact that the (conditioned) random walk is unlikely to travel a distance of $|x-y|$ in time n. The Brownian motion analogue of (71) was analyzed in Lemma 18 of [15], while we are unaware of general tight asymptotics for the random walk case. A notable exception is Theorem 1.2 of [20], which provided precise asymptotics of the ballot probability (71) for $x \in [0, n^{1/2-\varepsilon}]$, $y \in [C_1\sqrt{n}, C_2\sqrt{n \log n}]$ for fixed constants $C_1, C_2 > 0$. On the contrary, we satisfy ourselves with asymptotic upper bounds, which allow for a wider range of the parameter y. We first record below the result from [20] that we will employ.

Lemma 25. *Fix $\varepsilon \in (0, 1/2)$ and $L > 0$. There exists $\delta > 0$ such that uniformly for $x \in [0, n^{1/2-\varepsilon}]$ and $y \geqslant \sqrt{n}/L$,*

$$\mathbb{P}(x + S_n \in [y, y + 1],\ x + S_n \geqslant 0 \text{ for all } 1 \leqslant k \leqslant n) \ll xn^{-1-\varepsilon} + xyn^{-3/2}e^{-\frac{\delta y^2}{n}}.$$

Proof. By (3.1) of Theorem 3.1 in [20] (and following the beginning of the proof of Theorem 1.2 therein), it holds uniformly for $x \in [0, n^{1/2-\varepsilon}]$ that

$$\mathbb{P}(x + S_n \in [y, y + 1],\ x + S_n \geqslant 0 \text{ for all } 1 \leqslant k \leqslant n) \ll \frac{x}{n} J_n, \tag{72}$$

where

$$J_n := \int_{-\varepsilon}^{1+\varepsilon} \left(\left(1 + \frac{t + y}{\sqrt{n}}\right) e^{-\frac{\delta(t+y)^2}{n}} + n^{-\varepsilon}\right) \mathrm{d}t$$

for some large constant $C > 0$ depending on the law of the jump and ε. We have uniformly for $y \geqslant \sqrt{n}/L$,

$$J_n \ll n^{-\varepsilon} + \frac{y}{\sqrt{n}} e^{-\frac{\delta y^2}{n}}.$$

Combined with (72) finishes the proof. ☐

Let us now consider a logarithmically tilted barrier. Applying the same arguments that derived Lemma 23 from Lemma 24, and using Lemma 25 instead of Lemma 24, we arrive at the following result.

Lemma 26. *Fix $\varepsilon \in (0, 1/2)$ and $L, L' > 0$. There exists $\delta > 0$ such that uniformly for $x \in [0, n^{1/2-\varepsilon}]$ and $y \geqslant \sqrt{n}/L$,*

$$\mathbb{P}(x + S_n \in [y, y + 1],\ x + S_n \geqslant -L'(\log \min\{k, n - k\})_+ \text{ for all } 1 \leqslant k \leqslant n) \ll xn^{-1-\varepsilon} + xyn^{-3/2}e^{-\frac{\delta y^2}{n}}.$$

Due to the term $xn^{-1-\varepsilon}$, the bound in Lemma 26 cannot be tight for $y \gg \sqrt{n \log n}$. While a general tight bound seems reminiscent in the literature, the following weaker estimate suffices for our purpose.

Lemma 27. *Fix $L' > 0$. There exists $\delta > 0$ such that uniformly for $x, y > 0$,*

$$\mathbb{P}(x + S_n \in [y, y + 1],\ x + S_n \geqslant -L'(\log \min\{k, n - k\})_+ \text{ for all } 1 \leqslant k \leqslant n) \ll \varphi_{n,\delta}(|x - y|).$$

In particular, for $x \in [0, n^{1/2-\varepsilon}]$ with $\varepsilon \in (0, 1/2)$ and $y \gg \sqrt{n}$,

$$\mathbb{P}(x + S_n \in [y, y + 1],\ x + S_n \geqslant -L'(\log \min\{k, n - k\})_+ \text{ for all } 1 \leqslant k \leqslant n) \leqslant \mathbb{P}(|S_n| \geqslant |x - y|) \ll \varphi_{n,\delta}(y).$$

Proof. By a moderate deviation estimate (e.g., Theorem 3.7.1 of [14]), we have

$$\mathbb{P}(x + S_n \in [y, y + 1],\ x + S_n \geqslant -L'(\log \min\{k, n - k\})_+ \text{ for all } 1 \leqslant k \leqslant n) \leqslant \mathbb{P}(|S_n| \geqslant |x - y|) \ll \varphi_{n,\delta}(|x - y|)$$

for some $\delta > 0$. ☐

Remark 9. By symmetry, the same statements of Lemmas 26 and 27 hold with the roles of x, y interchanged. Moreover, the same results hold for $0 \leqslant x \ll n^{1/2-\varepsilon}$ by a suitable scaling.

C.3 BRW conditioned on two descendants with large displacements separated at the first step

In practice, when conditioning on the event $\mathscr{B}_h$ and a fixed location at time $t_x - h$, we would like to understand the conditional law of the BRW in the period $[t_x - h, t_{x,K}]$, given the information that two trajectories separated at time $t_x - h$ both reach the level x at time $t_{x,K}$. In this self-contained appendix, we consider a large number n and $g < 0$, and recall from (30) that

$$\mathscr{I}_{n,g} = \Big\{\exists\, v, w \in V_n,\ \mathrm{lca}(v, w) = \emptyset,\ \eta_{v,n}(n) \geqslant m_n - g,\ \eta_{w,n}(n) \geqslant m_n - g\Big\},\ n \in \mathbb{N},\ g < 0.$$

The general strategy to deal with an event of this type is to condition on the first generation and partition the event into disjoint events where at least two of them have descendants with large maxima. When conditioned on the first generation, the events of having large maxima will then be independent and hence can be decoupled.

In the following, for $j \in \mathbb{N}$, let $\mathbb{P}_j$ denote the law of the tuple $(\eta_1, \ldots, \eta_j)$ of i.i.d. random variables with law ξ. Throughout, assume (A1)–(A4).

Lemma 28. *For $g < 0$, it holds that $\mathbb{P}(\mathscr{I}_{n,g}) \asymp (|g|+1)^2 e^{2c_2 g}$.*

Proof. We condition on the first generation and obtain

$$\mathbb{P}(\mathscr{I}_{n,g}) = \sum_{j=2}^{\infty} p_j \int \mathbb{P}(\mathscr{I}_{n,g} \mid V_1 = \{v_k\}_{1 \leqslant k \leqslant j},\, \eta_{v_k,1}(1) = \eta_k)\, \mathrm{d}\mathbb{P}_j(\eta_1, \ldots, \eta_j).$$

We first prove the upper bound of $\mathbb{P}(\mathscr{I}_{n,g})$. By assumption (A4),

$$\mathbb{P}(\xi > x) \ll e^{-(c_2+\delta)x}. \tag{73}$$

For a fixed j, there are $\leqslant j^2$ possibilities of pairs (k,k'), $1 \leqslant k < k' \leqslant j$ so that descendants of v_k, $v_{k'}$ realize the event $\mathscr{I}_{n,g}$. By a union bound, we have

$$\begin{aligned}
&\int \mathbb{P}(\mathscr{I}_{n,g} \mid V_1 = \{v_k\}_{1 \leqslant k \leqslant j},\, \eta_{v_k,1}(1) = \eta_k)\, \mathrm{d}\mathbb{P}_j(\eta_1, \ldots, \eta_j) \\
&\leqslant j^2 \int \mathbb{P}(M_{n-1} > m_n - g - \eta_1)\mathbb{P}(M_{n-1} > m_n - g - \eta_2)\, \mathrm{d}\mathbb{P}_2(\eta_1, \eta_2) \\
&\ll j^2 \mathbb{E}\big[\min\{(|g+\xi|+1)e^{c_2(g+\xi)}, 1\}\big]^2 \\
&\ll j^2 (|g|+1)^2 e^{2c_2 g}.
\end{aligned}$$

where we have used (73) and Lemma 4. By assumption (A1), we conclude that

$$\mathbb{P}(\mathscr{I}_{n,g}) \ll \sum_{j=2}^{\infty} p_j j^2 (|g|+1)^2 e^{2c_2 g} \ll (|g|+1)^2 e^{2c_2 g},$$

as desired.

To show the lower bound of $\mathbb{P}(\mathscr{I}_{n,g})$, suppose that $p_j > 0$ for some $j \geqslant 2$ (valid since $\rho > 1$). Since the probability that j i.i.d. samples of η_1 are all positive is $\gg 1$, we have by (14) of Lemma 4,

$$\int \mathbb{P}(\mathscr{I}_{n,g} \mid V_1 = \{v_k\}_{1 \leqslant k \leqslant j},\, \eta_{v_k,1}(1) = \eta_k)\, \mathrm{d}\mathbb{P}_j(\eta_1, \ldots, \eta_j) \gg (|g|+1)^2 e^{2c_2 g},$$

as desired. ☐

Recall the defining barrier (33) for the ballot event $\mathscr{E}_{1,\ell}$. Let $V_{n,g}$ denote the set of particles $v \in V_n$ whose past trajectory $\{\eta_{v,n}(k)\}_{1 \leqslant k \leqslant n}$ does not cross the barrier

$$\widetilde{\psi}(k) := L \log n - g + \frac{k}{n} m_n + \frac{4}{c_2} (\log \min\{k, n-k\})_+, \ 1 \leqslant k \leqslant n.$$

Recall that $\varphi_{n,\delta}(i) := e^{-\delta |i| \min(\frac{|i|}{n}, 1)}$.

Lemma 29. *It holds that for $g < 0$,*

$$\mathbb{P}(\exists v \in V_{n,g},\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n - g, \mathbf{u})} \mid \mathscr{I}_{n,g}) \ll \begin{cases} n^{3/2} \varphi_{n,\delta}(\|\mathbf{u}\|) & \text{if } \|\mathbf{u}\| > \sqrt{n} \log n; \\ (\log n)^2 n^{-\frac{d-1}{2}} & \text{if } \|\mathbf{u}\| \leqslant \sqrt{n} \log n. \end{cases} \tag{74}$$

Proof. We write

$$\mathbb{P}(\exists v \in V_{n,g},\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n - g, \mathbf{u})} \mid \mathscr{I}_{n,g}) = \frac{\mathbb{P}(\exists v \in V_{n,g},\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n - g, \mathbf{u})},\, \mathscr{I}_{n,g})}{\mathbb{P}(\mathscr{I}_{n,g})}.$$

The denominator is bounded from above by $\mathbb{P}(\mathscr{I}_{n,g}) \gg (|g|+1)^2 e^{2c_2 g}$ by Lemma 28, and hence it suffices to establish an upper bound for the numerator. To this end, we condition on the law of the first generation, by first conditioning on $\#V_1$ and then the locations of particles belonging to the first generation, we have

$$\begin{aligned}
&\mathbb{P}(\exists v \in V_{n,g},\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n-g,\mathbf{u})},\, \mathscr{I}_{n,g}) \\
&= \sum_{j=2}^{\infty} p_j \int \mathbb{P}(\exists v \in V_{n,g},\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n-g,\mathbf{u})},\, \mathscr{I}_{n,g} \mid V_1 = \{v_k\}_{1\leqslant k\leqslant j},\, \boldsymbol{\eta}_{v_k,1}(1) = \boldsymbol{\eta}_k)\, \mathrm{d}\mathbb{P}_j(\boldsymbol{\eta}_1,\dots,\boldsymbol{\eta}_j).
\end{aligned}$$

To deal with the inner probability, we decompose into sub-events where the event $\mathscr{I}_{n,g}$ is realized by descendants of v_k and $v_{k'}$ (which we denote by $\mathscr{I}_{n,g}^{k,k'}$), $1 \leqslant k < k' \leqslant j$, and decompose the count based on the ancestor at time one. We obtain

$$\begin{aligned}
&\mathbb{P}(\exists v \in V_{n,g},\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n-g,\mathbf{u})},\, \mathscr{I}_{n,g} \mid V_1 = \{v_k\}_{1\leqslant k\leqslant j},\, \boldsymbol{\eta}_{v_k,1}(1) = \boldsymbol{\eta}_k) \\
&\leqslant \sum_{1\leqslant k<k'\leqslant j} \sum_{i=1}^{j} \mathbb{P}(\exists v \in V_{n,g},\, v \succ v_i,\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n-g,\mathbf{u})},\, \mathscr{I}_{n,g}^{k,k'} \mid V_1 = \{v_k\}_{1\leqslant k\leqslant j},\, \boldsymbol{\eta}_{v_k,1}(1) = \boldsymbol{\eta}_k).
\end{aligned}$$

We may then apply independence to bound the probability. In the case $i = k$,

$$\begin{aligned}
&\mathbb{P}(\exists v \in V_{n,g},\, v \succ v_i,\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n-g,\mathbf{u})},\, \mathscr{I}_{n,g}^{k,k'} \mid V_1 = \{v_k\}_{1\leqslant k\leqslant j},\, \boldsymbol{\eta}_{v_k,1}(1) = \boldsymbol{\eta}_k) \\
&\leqslant \mathbb{P}(M_{n-1} > m_n - g - \eta_{k'})\mathbb{P}(\exists v \in V_{n-1,g},\, \boldsymbol{\eta}_{v,n-1}(n-1) \in B_{(m_n-g,\mathbf{u})-\boldsymbol{\eta}_k}) \\
&\ll \min\{(|g+\eta_{k'}|+1)e^{c_2(g+\eta_{k'})}, 1\}\mathbb{P}(\exists v \in V_{n-1,g},\, \boldsymbol{\eta}_{v,n-1}(n-1) \in B_{(m_n-g,\mathbf{u})-\boldsymbol{\eta}_k}),
\end{aligned}$$

where we have used Lemma 4. Suppose that $\|\mathbf{u}\| \leqslant \sqrt{n}\log n$. By a change of measure argument and Lemma 23,

$$\mathbb{P}(\exists v \in V_{n-1,g},\, \boldsymbol{\eta}_{v,n-1}(n-1) \in B_{(m_n-g,\mathbf{u})-\boldsymbol{\eta}_k}) \ll (\log n)(\log n - g)e^{c_2(g+\eta_k)} n^{-\frac{d-1}{2}}. \tag{75}$$

We therefore conclude that for $\|\mathbf{u}\| \leqslant \sqrt{n}\log n$,

$$\begin{aligned}
&\mathbb{P}(\exists v \in V_{n,g},\, v \succ v_i,\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n-g,\mathbf{u})},\, \mathscr{I}_{n,g}^{k,k'} \mid V_1 = \{v_k\}_{1\leqslant k\leqslant j},\, \boldsymbol{\eta}_{v_k,1}(1) = \boldsymbol{\eta}_k) \\
&\ll (\log n)^2(|g|+1)e^{c_2(g+\eta_k)} \min\{(|g+\eta_{k'}|+1)e^{c_2(g+\eta_{k'})}, 1\} n^{-\frac{d-1}{2}}.
\end{aligned}$$

The case $i = k'$ is similar. In the case $i \notin \{k, k'\}$ and $\|\mathbf{u}\| \leqslant \sqrt{n}\log n$, by Lemma 4 and Lemma 23,

$$\begin{aligned}
&\mathbb{P}(\exists v \in V_{n,g},\, v \succ v_i,\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n-g,\mathbf{u})},\, \mathscr{I}_{n,g}^{k,k'} \mid V_1 = \{v_k\}_{1\leqslant k\leqslant j},\, \boldsymbol{\eta}_{v_k,1}(1) = \boldsymbol{\eta}_k) \\
&\leqslant \mathbb{P}(M_{n-1} > m_n - g - \eta_k)\mathbb{P}(M_{n-1} > m_n - g - \eta_{k'})\mathbb{P}(\exists v \in V_{n-1,g},\, \boldsymbol{\eta}_{v,n-1}(n-1) \in B_{(m_n-g,\mathbf{u})-\boldsymbol{\eta}_i}) \\
&\ll \min\{(|g+\eta_k|+1)e^{c_2(g+\eta_k)}, 1\} \min\{(|g+\eta_{k'}|+1)e^{c_2(g+\eta_{k'})}, 1\}(\log n)(\log n - g)e^{c_2(g+\eta_i)} n^{-\frac{d-1}{2}} \\
&\ll (\log n)^2(|g|+1)e^{c_2(g+\eta_i)} n^{-\frac{d-1}{2}} \min\{(|g+\eta_k|+1)e^{c_2(g+\eta_k)}, 1\} \min\{(|g+\eta_{k'}|+1)e^{c_2(g+\eta_{k'})}, 1\}.
\end{aligned}$$

Combining the above, we have by (73) that for the case $\|\mathbf{u}\| \leqslant \sqrt{n}\log n$,

$$\begin{aligned}
&\mathbb{P}(\exists v \in V_{n,g},\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n-g,\mathbf{u})},\, \mathscr{I}_{n,g}) \\
&\ll \sum_{j=2}^{\infty} p_j n^{-\frac{d-1}{2}} \Big(j^2(\log n)^2(|g|+1)e^{c_2 g}\mathbb{E}\big[\min\{|g+\xi|e^{c_2(g+\xi)}, 1\}\big]\mathbb{E}[e^{c_2\xi}] \\
&\qquad\qquad + j^3(\log n)^2(|g|+1)e^{c_2 g}\mathbb{E}\big[\min\{|g+\xi|e^{c_2(g+\xi)}, 1\}\big]^2\mathbb{E}[e^{c_2\xi}]\Big) \\
&\ll n^{-\frac{d-1}{2}}(\log n)^2(|g|+1)^2 e^{2c_2 g}\sum_{j=2}^{\infty} p_j j^2 + n^{-\frac{d-1}{2}}(\log n)^2(|g|+1)^3 e^{3c_2 g}\sum_{j=2}^{\infty} p_j j^3 \\
&\ll n^{-\frac{d-1}{2}}(\log n)^2(|g|+1)^2 e^{2c_2 g},
\end{aligned}$$

where we use assumption (A1) in the last step. This proves (74) in the case $\|\mathbf{u}\| \leqslant \sqrt{n}\log n$.

In the case $\|\mathbf{u}\| > \sqrt{n}\log n$, we need to replace (75) accordingly. Applying a change of measure as in the proof of Lemma 14,

$$\begin{aligned}
&\mathbb{P}(\exists v \in V_{n-1,g},\, \boldsymbol{\eta}_{v,n-1}(n-1) \in B_{(m_n-g,\mathbf{u})-\boldsymbol{\eta}_k}) \\
&\ll n^{3/2} e^{c_2(g+\eta_k)} \mathbb{P}(\mathbf{S}_{n-1} \in B_{(m_n-g,\mathbf{u})-\boldsymbol{\eta}_k},\, S_k \leqslant \widetilde{\psi}(k) \text{ for all } 1 \leqslant k \leqslant n-1) \\
&\ll n^{3/2} e^{c_2(g+\eta_k)} \mathbb{P}(\|\widehat{\mathbf{S}}_{n-1} - (\mathbf{u} - \widehat{\boldsymbol{\eta}}_k)\| \leqslant 1) \\
&\ll n^{3/2} e^{c_2(g+\eta_k)} \varphi_{n,\delta}(\|\mathbf{u} - \widehat{\boldsymbol{\eta}}_k\|).
\end{aligned}$$

We apply the bound $\varphi_{n,\delta}(\|\mathbf{u}-\widehat{\boldsymbol{\eta}}_k\|) \leqslant \varphi_{n,\delta}(\|\mathbf{u}\|/2)$ on the event $\|\widehat{\boldsymbol{\eta}}_k\| \leqslant \|\mathbf{u}\|/2$, and apply the bound (75) on the complement of this event which still occurs with probability $\ll \varphi_{n,\delta}(\|\mathbf{u}\|/2)$ by (73). More precisely, for the case $i = k$, we use

$$\begin{aligned}
&\mathbb{P}(\exists v \in V_{n,g},\, v \succ v_i,\, \boldsymbol{\eta}_{v,n}(n) \in B_{(m_n-g,\mathbf{u})},\, \mathscr{I}_{n,g}^{k,k'} \mid V_1 = \{v_k\}_{1\leqslant k\leqslant j},\, \boldsymbol{\eta}_{v_k,1}(1) = \boldsymbol{\eta}_k) \\
&\ll \min\{(|g+\eta_{k'}|+1)e^{c_2(g+\eta_{k'})}, 1\} \times \begin{cases} n^{3/2} e^{c_2(g+\eta_k)} \varphi_{n,\delta}(\|\mathbf{u}\|/2) & \text{if } \|\widehat{\boldsymbol{\eta}}_k\| \leqslant \|\mathbf{u}\|/2; \\ (\log n)(\log n - g) e^{c_2(g+\eta_k)} n^{-\frac{d-1}{2}} & \text{if } \|\widehat{\boldsymbol{\eta}}_k\| > \|\mathbf{u}\|/2. \end{cases}
\end{aligned}$$

This part of the sum is then controlled by

$$\begin{aligned}
&\sum_{j=2}^{\infty} p_j j^2 \Big(n^{-\frac{d-1}{2}} (\log n)^2 (|g|+1) e^{c_2 g} \mathbb{E}\big[\min\{|g+\xi| e^{c_2(g+\xi)}, 1\}\big] \mathbb{E}[e^{c_2\xi} \mathbb{1}_{\{\|\widehat{\boldsymbol{\eta}}_k\| > \|\mathbf{u}\|/2\}}] \\
&\qquad\qquad + n^{3/2} e^{c_2(g+\eta_k)} \varphi_{n,\delta}\big(\frac{\|\mathbf{u}\|}{2}\big) \mathbb{E}\big[\min\{|g+\xi| e^{c_2(g+\xi)}, 1\}\big]\Big) \\
&\ll n^{3/2} \varphi_{n,\delta}(\|\mathbf{u}\|)(|g|+1)^2 e^{2c_2 g},
\end{aligned}$$

where the δ may vary from line to line. The rest of the argument follows analogously as the case $\|\mathbf{u}\| \leqslant \sqrt{n}\log n$. ☐

D A conditional local CLT (for general jumps)

In this appendix, we prove a more general version of Lemma 9, which deals with a general increment distribution $\boldsymbol{\xi}$ that may not be spherically symmetric (see the discussion in Section 4). The result is indeed an application of Petrov's theorem [37] and a change of measure argument, while we include the details for completeness. Recall the setting below (3) of the law $\boldsymbol{\xi}$. If $\boldsymbol{\xi}$ is spherically symmetric, we have $\widehat{c}_1 = c_1$ and $\mathbf{c}_2 = c_2\mathbf{e}_1$.

Define the set

$$\widetilde{B}_{\mathbf{c}_2}(x;r) := \Big\{\mathbf{x} + s\mathbf{c}_2 + \mathbf{y} : \mathbf{y}\cdot\mathbf{c}_2 = 0,\, \|\mathbf{y}\| \leqslant r,\, |s| \leqslant r\Big\},\ r > 0$$

and (for this appendix only)

$$\widetilde{t}_x := \frac{x}{\widehat{c}_1} + \frac{d+2}{2\widehat{c}_1 \partial_{x_1}\widehat{I}(\widehat{c}_1,\mathbf{0})} \log x - (\log x)^2.$$

Lemma 9 then follows from the next result.[13]

Lemma 30 (conditional local CLT)**.** *Fix $L, r > 0$. Uniformly for $\boldsymbol{\lambda}(x) = O((\log x)^L)$,*

$$\mathbb{P}\Big(\boldsymbol{\lambda}(x) + \mathbf{S}_{\widetilde{t}_x} \in \widetilde{B}_{\mathbf{c}_2}(x;r) \mid (\boldsymbol{\lambda}(x) + \mathbf{S}_{\widetilde{t}_x})\cdot\mathbf{c}_2 \in [x\mathbf{c}_2\cdot\mathbf{e}_1 - r, x\mathbf{c}_2\cdot\mathbf{e}_1 + r]\Big) \asymp x^{-\frac{d-1}{2}}.$$

Proof. Let $\Lambda(\boldsymbol{\lambda}) = \log\phi_{\boldsymbol{\xi}}(\boldsymbol{\lambda}) = \log\mathbb{E}[e^{\boldsymbol{\lambda}\cdot\boldsymbol{\xi}}]$ be the log-moment generating function of $\boldsymbol{\xi}$. The measure $\mathbb{Q}$ defined by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathbf{x}) := e^{\mathbf{c}_2\cdot\mathbf{x} - \Lambda(\mathbf{c}_2)} \tag{76}$$

satisfies that under $\mathbb{Q}$, $\{\boldsymbol{\xi}_i\}_{i\in\mathbb{N}}$ are i.i.d. with mean $(\widehat{c}_1,\mathbf{0}) \in \mathbb{R}^d$. In other words, under $\mathbb{Q}$, the random walk $\{\widetilde{\mathbf{S}}_n\} := \{\mathbf{S}_n - n(\widehat{c}_1,\mathbf{0})\}$ is centered.

[13] When applying the next result to prove Theorem 2, one picks r small enough (depending only on $\mathbf{c}_2$) such that $\widetilde{B}_{\mathbf{c}_2}(x;r) \subseteq B_x$.

By assumption (A5), the law of $\boldsymbol{\xi}$ under $\mathbb{P}$ is non-lattice. It follows by definition and triangle inequality that the law of $\boldsymbol{\xi}$ (and hence of its projection $\boldsymbol{\xi}\cdot\mathbf{c}_2$) under $\mathbb{Q}$ is also non-lattice. By the local CLT,

$$\begin{aligned}
&\mathbb{P}\Big((\boldsymbol{\lambda}(x)+\mathbf{S}_{\widetilde{t}_x})\cdot\mathbf{c}_2\in[x\mathbf{c}_2\cdot\mathbf{e}_1-r,x\mathbf{c}_2\cdot\mathbf{e}_1+r]\Big)\\
&\asymp e^{\widetilde{t}_x\Lambda(\mathbf{c}_2)-(x\mathbf{e}_1-\boldsymbol{\lambda}(x))\cdot\mathbf{c}_2}\mathbb{Q}\Big((\boldsymbol{\lambda}(x)+\mathbf{S}_{\widetilde{t}_x})\cdot\mathbf{c}_2\in[x\mathbf{c}_2\cdot\mathbf{e}_1-r,x\mathbf{c}_2\cdot\mathbf{e}_1+r]\Big)\\
&\asymp e^{\widetilde{t}_x\Lambda(\mathbf{c}_2)-(x\mathbf{e}_1-\boldsymbol{\lambda}(x))\cdot\mathbf{c}_2}\mathbb{Q}\Big(\Big|\widetilde{\mathbf{S}}_{\widetilde{t}_x}\cdot\mathbf{c}_2-\Big((\widehat{c}_1(\log x)^2-\frac{d+2}{2\partial_{x_1}\widehat{I}(\widehat{c}_1,\mathbf{0})}\log x)\,\mathbf{e}_1\cdot\mathbf{c}_2-\boldsymbol{\lambda}(x)\cdot\mathbf{c}_2\Big)\Big|\leqslant r\Big)\\
&\asymp e^{\widetilde{t}_x\Lambda(\mathbf{c}_2)-(x\mathbf{e}_1-\boldsymbol{\lambda}(x))\cdot\mathbf{c}_2}x^{-\frac{1}{2}}.
\end{aligned}$$

Similarly, using the multi-dimensional local CLT,

$$\begin{aligned}
\mathbb{P}\Big(\boldsymbol{\lambda}(x)+\mathbf{S}_{\widetilde{t}_x}\in\widetilde{B}_{\mathbf{c}_2}(x;r)\Big)&\asymp e^{\widetilde{t}_x\Lambda(\mathbf{c}_2)-(x\mathbf{e}_1-\boldsymbol{\lambda}(x))\cdot\mathbf{c}_2}\mathbb{Q}\Big(\boldsymbol{\lambda}(x)+\mathbf{S}_{\widetilde{t}_x}\in\widetilde{B}_{\mathbf{c}_2}(x;r)\Big)\\
&\asymp e^{\widetilde{t}_x\Lambda(\mathbf{c}_2)-(x\mathbf{e}_1-\boldsymbol{\lambda}(x))\cdot\mathbf{c}_2}x^{-\frac{d}{2}}.
\end{aligned}$$

This completes the proof. □