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A probabilistic approach to the Φ-variation of classical fractal functions with critical roughness[☆]

Xiyue Han, Alexander Schied [*], Zhenyuan Zhang

Department of Statistics and Actuarial Science, University of Waterloo, Canada

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ABSTRACT

We consider Weierstraß and Takagi–van der Waerden functions with critical degree of roughness. In this case, the functions have vanishing pth variation for all $p > 1$ but are also nowhere differentiable and hence not of bounded variation either. We resolve this apparent puzzle by showing that these functions have finite, nonzero, and linear Wiener–Young Φ-variation along the sequence of b-adic partitions, where $\Phi(x) = x/\sqrt{-\log x}$. For the Weierstraß functions, our proof is based on the martingale central limit theorem (CLT). For the Takagi–van der Waerden functions, we use the CLT for Markov chains if a certain parameter b is odd, and the standard CLT for b even.

1. Introduction and statement of results

We consider a base function $\varphi : \mathbb{R} \to \mathbb{R}$ that is periodic with period 1 and Lipschitz continuous. Our aim is to study the function

$$f(t) := \sum_{m=0}^{\infty} \alpha^m \varphi(b^m t), \qquad t \in [0, 1], \tag{1.1}$$

where $b \in \{2, 3, \dots\}$ and $\alpha \in (-1, 1)$. Then the series on the right-hand side converges absolutely and uniformly in $t \in [0, 1]$, so that f is indeed a well defined continuous function. If $\varphi(t) = \nu \sin(2\pi t) + \rho \cos(2\pi t)$ for real constants ν and ρ, then f is a Weierstraß function. If $\varphi(t) = \min_{z\in\mathbb{Z}} |z - t|$ is the tent map, then f is a Takagi–van der Waerden function. It was shown in Schied and Zhang (2020) that, under some mild conditions on φ, the function f is of bounded variation for $|\alpha| < 1/b$, whereas for $|\alpha| > 1/b$ and $p := -\log_{|\alpha|} b$ it has nontrivial and linear pth variation along the sequence

$$\mathbb{T}_n := \{kb^{-n} : k = 0, \dots, b^n\}, \quad n \in \mathbb{N}, \tag{1.2}$$

of b-adic partitions of [0, 1]. That is, for all $t \in (0, 1]$,

$$\langle f\rangle_t^{(q)} := \lim_{n\uparrow\infty} \sum_{k=0}^{\lfloor tb^n \rfloor} \left|f((k+1)b^{-n}) - f(kb^{-n})\right|^q = \begin{cases} 0 & \text{if } q > p, \\ t \cdot \mathbb{E}[|Z|^q] & \text{if } q = p, \\ +\infty & \text{if } q < p. \end{cases} \tag{1.3}$$

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* Corresponding author.
E-mail addresses: xiyue.han@uwaterloo.ca (X. Han), aschied@uwaterloo.ca (A. Schied), zhenyuan.zhang@uwaterloo.ca (Z. Zhang).

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Here, Z is a certain random variable, whose law is known in some special cases. For instance, if φ is the tent map and b is even, then the law of αbZ is the infinite Bernoulli convolution with parameter $1/(|\alpha|b)$ (see also Gantert, 1994; Schied, 2016; Mishura and Schied, 2019 for earlier results in this special setup). Clearly, the parameter $p = -\log_{|\alpha|} b$ can be regarded as a measure for the "roughness" of the function f. As a matter of fact, it is well known that a typical sample path $t \mapsto B_H(t)$ of a fractional Brownian motion has linear pth variation $\langle B_H\rangle_t^{(p)} = t\cdot\mathbb{E}[|B^H(1)|^p]$ for $p = 1/H$.

Remark 1.1 (*On the connection with pathwise Itô calculus*). Our interest in the pth variation of fractal functions is motivated by its connection to pathwise Itô calculus. For instance, if $|\alpha| = 1/\sqrt{b}$, we have $p = 2$ and the limit in (1.3) is just the usual quadratic variation of the function f, taken along the partition sequence $\{\mathbb{T}_n\}_{n\in\mathbb{N}}$. It was observed by Föllmer (1981) that the existence of this limit is sufficient for the validity of Itô's formula with integrator f, and this is the key to a rich theory of pathwise Itô calculus with applications to robust finance; see, e.g., Föllmer and Schied (2013) for a discussion. Recently, Cont and Perkowski (2019) extended Föllmer's Itô formula to functions with finite pth variation, which has led to a substantial increase in the interest in corresponding "rough" trajectories with $p > 2$.

In this note, we study the case of critical roughness, $\alpha = -1/b$ or $\alpha = 1/b$, in which $p = 1$. For this case, it was shown in Schied and Zhang (2020) that $\langle f\rangle_t^{(q)} = 0$ for all $q > 1$ and $t \in [0, 1]$. This, however, does *not* imply that f is of bounded variation. For instance, if φ is the tent map, $b = 2$, and $\alpha = 1/2$, then f is the classical Takagi function, which is nowhere differentiable and hence cannot be of bounded variation; a very short proof of this fact was given by de Rham (1957) and later rediscovered by Billingsley (1982). For the Weierstraß function, the proof of nowhere differentiability for all $\alpha \in [1/b, 1)$ is more difficult. Starting from Weierstraß's original work, it attracted numerous authors until a definite result was given by Hardy (1916).

It is therefore apparent that, in the critical case $|\alpha| = 1/b$, power variation $\langle f\rangle^{(q)}$ is insufficient to capture the exact degree of roughness of the function f. To give a precise result on the roughness of the function f in the critical case, we take a strictly increasing function $\Phi : [0, 1) \to [0, \infty)$ and investigate the limit

$$\langle f\rangle_t^{\Phi} := \lim_{n\uparrow\infty} \sum_{k=0}^{\lfloor tb^n\rfloor} \Phi\big(|f((k+1)b^{-n}) - f(kb^{-n})|\big),$$

which can be regarded as the Wiener–Young Φ-variation of f (see, e.g., Appell et al., 2014), restricted to the sequence of b-adic partitions (1.2). Our main results will show that the correct choice for Φ is the function

$$\Phi(x) = \frac{x}{\sqrt{-\log x}} \quad \text{for } x \in (0, 1) \quad \text{and} \quad \Phi(0) := 0.$$

We fix this function Φ throughout the remainder of this paper. Our first result establishes the Φ-variation of f from (1.1) for the class of Takagi–van der Waerden functions.

Theorem 1.2. *Let $\varphi(t) = \min_{z\in\mathbb{Z}} |z - t|$ be the tent map, $b \in \{2, 3, \dots\}$, and $|\alpha| = 1/b$. Then the Φ-variation of the Takagi–van der Waerden function f exists along $\{\mathbb{T}_n\}_{n\in\mathbb{N}}$. If b is even, then it is given by*

$$\langle f\rangle_t^{\Phi} = t\cdot\sqrt{\frac{2}{\pi\log b}}, \qquad t \in [0, 1].$$

If b is odd, then

$$\langle f\rangle_t^{\Phi} = t\cdot\sqrt{\frac{2(b + \operatorname{sgn}(\alpha))}{\pi(b - \operatorname{sgn}(\alpha))\log b}}, \qquad t \in [0, 1].$$

Our results will be consequences of suitable central limit theorems (CLTs). In the preceding theorem, the case of b even will be settled by the standard CLT, whereas the case of b odd will require the use of a CLT for Markov chains. For establishing the Φ-variation of the critical Weierstraß functions, as stated in the following theorem, we rely on the martingale CLT. A loosely related CLT for the classical Takagi function was proved by Gamkrelidze (1990).

Theorem 1.3. *Suppose $\varphi(t) = \nu\sin(2\pi t) + \rho\cos(2\pi t)$, $b \in \{2, 3, \dots\}$, and $|\alpha| = 1/b$. Then the Φ-variation of the Weierstraß function f exists along $\{\mathbb{T}_n\}_{n\in\mathbb{N}}$ and is given by*

$$\langle f\rangle_t^{\Phi} = t\cdot 2\sqrt{\frac{\pi(\nu^2 + \rho^2)}{\log b}}, \qquad t \in [0, 1].$$

2. Proofs

We first consider only the Φ-variation $\langle f\rangle_t^{\Phi}$ for $t = 1$. The case $t < 1$ will be discussed at the end of this section, simultaneously for both theorems. We fix $b \in \{2, 3, \dots\}$ and $\alpha \in \{-1/b, +1/b\}$. Following Schied and Zhang (2020), we let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space supporting an independent sequence $U_1, U_2, \ldots$ of random variables with a uniform distribution on $\{0, 1, \ldots, b-1\}$ and define the stochastic process $R_m := \sum_{i=1}^{m} U_i b^{i-1}$. Note that R_m has a uniform distribution on $\{0, \ldots, b^m - 1\}$. Therefore, for $n \in \mathbb{N}$ such that all increments $\big|f((k+1)b^{-n}) - f(kb^{-n})\big|$ are less than 1,

$$V_n := \sum_{k=0}^{b^n-1} \Phi\Big(\big|f((k+1)b^{-n}) - f(kb^{-n})\big|\Big) = b^n \mathbb{E}\Big[\Phi\Big(\big|f((R_n+1)b^{-n}) - f(R_n b^{-n})\big|\Big)\Big]. \tag{2.1}$$

To simplify the expectation on the right, let the nth truncation of f be given by $f_n(t) = \sum_{m=0}^{n-1} \alpha^m \varphi(b^m t)$. The periodicity of φ implies that

$$\begin{aligned} f((R_n+1)b^{-n}) - f(R_n b^{-n}) &= f_n((R_n+1)b^{-n}) - f_n(R_n b^{-n}) \\ &= b^{-n}\mathrm{sgn}(\alpha)^n \sum_{m=1}^{n} \mathrm{sgn}(\alpha)^m \frac{\varphi((R_n+1)b^{-m}) - \varphi(R_n b^{-m})}{b^{-m}}. \end{aligned}$$

The periodicity of φ implies moreover that for $m \le n$,

$$\varphi\big(x + R_n b^{-m}\big) = \varphi\Big(x + \sum_{i=1}^{n} U_i b^{i-1-m}\Big) = \varphi\Big(x + \sum_{i=1}^{m} U_i b^{i-1-m}\Big) = \varphi\big(x + R_m b^{-m}\big).$$

Therefore,

$$\mathrm{sgn}(\alpha)^m \frac{\varphi((R_n+1)b^{-m}) - \varphi(R_n b^{-m})}{b^{-m}} = \mathrm{sgn}(\alpha)^m \frac{\varphi((R_m+1)b^{-m}) - \varphi(R_m b^{-m})}{b^{-m}} =: Y_m.$$

It follows that

$$V_n = b^n \mathbb{E}\Big[\Phi\Big(b^{-n}\Big|\sum_{m=1}^{n} Y_m\Big|\Big)\Big]. \tag{2.2}$$

Lemma 2.1. *Suppose that $Z_0, Z_1, Z_2, \ldots$ is a sequence of random variables with $Z_0 = 0$ and uniformly bounded increments such that the laws of $\frac{1}{\sqrt{n}} Z_n$ converge weakly to some normal distribution $N(0, \sigma^2)$ with $\sigma^2 > 0$ and that the expression $\frac{1}{n}\mathbb{E}[Z_n^2]$ is bounded in n. Then*

$$b^n \mathbb{E}\Big[\Phi\big(b^{-n}|Z_n|\big)\Big] \longrightarrow \sqrt{\frac{2\sigma^2}{\pi \log b}}.$$

Proof. The fact that $\frac{1}{n}\mathbb{E}[Z_n^2]$ is bounded implies together with standard arguments that for every nondegenerate interval $I \subset [0, \infty)$,

$$\lim_{n\uparrow\infty} \mathbb{E}\Big[\mathbb{1}_{\{|\frac{1}{\sqrt{n}}Z_n| \in I\}} \Big|\frac{1}{\sqrt{n}} Z_n\Big|\Big] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\{|z| \in I\}} |z| e^{-z^2/(2\sigma^2)}\, dz. \tag{2.3}$$

We have

$$b^n \mathbb{E}\Big[\Phi\big(b^{-n}|Z_n|\big)\Big] = \mathbb{E}\Big[\frac{|Z_n|}{\sqrt{n\log b - \log|Z_n|}} \mathbb{1}_{\{|Z_n|>0\}}\Big].$$

Let C be an almost sure uniform bound for $|Z_{k+1} - Z_k|$. Hence, for all $\beta \in (0, \log b)$ there exists $n_0 \in \mathbb{N}$ such that $n\beta < n\log b - \log(Cn)$ for all $n \ge n_0$. Hence,

$$\sqrt{n\log b - \log|Z_n|} \ge \sqrt{n\beta} \quad \text{for } n \ge n_0, \tag{2.4}$$

and taking $I := (0, \infty)$ in (2.3) gives

$$\limsup_{n\uparrow\infty} b^n \mathbb{E}\Big[\Phi\big(b^{-n}|Z_n|\big)\Big] \le \frac{1}{\sqrt{2\pi\sigma^2\beta}} \int_{\{|z| \in I\}} |z| e^{-z^2/(2\sigma^2)}\, dz = \sqrt{\frac{2\sigma^2}{\pi\beta}}.$$

To get a lower bound, observe that for every $\varepsilon > 0$ and $n \ge 1/\varepsilon^2$,

$$\mathbb{1}_{\{|\frac{1}{\sqrt{n}}Z_n| \ge \varepsilon\}} \sqrt{n\log b - \log|Z_n|} \le \mathbb{1}_{\{|\frac{1}{\sqrt{n}}|Z_n| \ge \varepsilon\}} \sqrt{n\log b}.$$

Hence, we get from (2.3) that

$$\liminf_{n\uparrow\infty} b^n \mathbb{E}\Big[\Phi\big(b^{-n}|Z_n|\big)\Big] \ge \frac{1}{\sqrt{2\pi\sigma^2\log b}} \int_{\{|z| \ge \varepsilon\}} |z| e^{-z^2/(2\sigma^2)}\, dz.$$

Sending $\varepsilon \downarrow 0$ and $\beta \uparrow \log b$ gives the result. $\square$

Proof of Theorem 1.2 for $t = 1$. For b even, Schied and Zhang (2020, Proposition 3.2 (a)) state that $Y_1, Y_2, \ldots$ is an i.i.d. sequence of symmetric $\{-1, +1\}$-valued Bernoulli random variables. Therefore, (2.2), the classical CLT, and Lemma 2.1 give $V_n \to \sqrt{2/(\pi \log b)}$. If b is odd, then Schied and Zhang (2020, Proposition 3.2 (b)) states that the random variables $\mathrm{sgn}(\alpha)^m Y_m$ form a time-homogeneous Markov chain on $\{-1, 0, +1\}$ with initial distribution $\mu_1 = (\frac{b-1}{2b}, \frac{1}{b}, \frac{b-1}{2b})$ and transition matrix P_+, where

$$P_\pm := \frac{1}{2b}\begin{pmatrix} b\pm 1 & 0 & b\mp 1 \\ b-1 & 2 & b-1 \\ b\mp 1 & 0 & b\pm 1\end{pmatrix}.$$

It follows that $Y_1, Y_2, \ldots$ also form a time-homogeneous Markov chain with initial distribution μ_1 and transition matrix P_+ for $\alpha > 0$ and P_- for $\alpha < 0$. Since 0 is a transient state, we can clearly consider only the restriction of the Markov chain to its positive recurrent states, -1 and $+1$. Let $\overline{P}_\pm$ be the 2×2-matrix obtained from $P_\pm$ by deleting the second row and second column from P, and define $\bar{\mu}_1 = (1/2, 1/2)$. Then $\bar{\mu}_1$ is the unique stationary distribution for $\overline{P}_\pm$. Moreover,

$$\overline{P}^n_\pm = \frac{1}{2}\begin{pmatrix} 1+(\pm b)^{-n} & 1-(\pm b)^{-n} \\ 1-(\pm b)^{-n} & 1+(\pm b)^{-n}\end{pmatrix}.$$

For the state-constraint Markov chain $\overline{Y}_1, \overline{Y}_2, \ldots$ with initial distribution $\bar{\mu}_1$ and transition matrix $\overline{P}_\pm$, we thus have $\mathrm{var}(\overline{Y}_1) = 1$ and

$$\mathrm{cov}(\overline{Y}_1, \overline{Y}_{n+1}) = \sum_{y_1, y_{n+1} \in \{-1,+1\}} \bar{\mu}_1(y_1)\overline{P}^n_\pm(y_1, y_{n+1}) y_1 y_{n+1} = (\pm b)^{-n}.$$

Letting

$$\sigma^2 := \mathrm{var}(\overline{Y}_1) + 2\sum_{n=1}^{\infty}\mathrm{cov}(\overline{Y}_1, \overline{Y}_{n+1}) = \frac{b\pm 1}{b\mp 1},$$

the central limit theorem for Markov chains (see, e.g., Jones, 2004) implies that $\frac{1}{\sqrt{n}}\sum_{k=1}^n \overline{Y}_k$ converges in law to $N(0, \sigma^2)$. Due to the stationarity of the Markov chain, we have moreover

$$\begin{aligned}\mathbb{E}\Big[\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^n \overline{Y}_k\Big)^2\Big] &= \frac{1}{n}\sum_{k=1}^n \mathrm{var}(\overline{Y}_k) + \frac{2}{n}\sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n}\mathrm{cov}(\overline{Y}_k, \overline{Y}_\ell) \\ &= 1 + \frac{2}{n}\sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n}(\pm b)^{k-\ell} \le 1 + \frac{2}{n}\cdot\frac{b^{1-n}+bn+b-n}{(b-1)^2},\end{aligned}$$

which is uniformly bounded in n. Therefore, Lemma 2.1 and (2.2) give $V_n \to \sqrt{2(b\pm 1)/(\pi(b\mp 1)\log b)}$. □

Now we prepare for the proof of Theorem 1.3 for $t = 1$. Let $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and $\mathscr{F}_n := \sigma(U_1, \ldots, U_n)$ for $n \in \mathbb{N}$. Then each Y_n is $\mathscr{F}_n$-measurable. Since $U_1, \ldots, U_n$ can be recovered from R_n, we have $\mathscr{F}_n = \sigma(R_n)$ for $n \ge 1$. We define $Z_0 := 0$ and $Z_n := \sum_{k=1}^n Y_k$ for $n \in \mathbb{N}$.

Lemma 2.2. *If $\varphi(t) = \nu\sin(2\pi t) + \rho\cos(2\pi t)$, then $\{Z_n\}_{n\in\mathbb{N}_0}$ is a martingale with respect to $\{\mathscr{F}_n\}_{n\in\mathbb{N}_0}$.*

Proof. We must show that $\mathbb{E}[Y_n|R_{n-1}] = 0$ $\mathbb{P}$-a.s. for $n \ge 1$. To this end, we use that $R_n = R_{n-1} + U_n b^{n-1}$, where $R_0 := 0$ and U_n is independent of R_{n-1}. Therefore,

$$\begin{aligned}\mathbb{E}[Y_n|R_{n-1} = r] &= (\mathrm{sgn}(\alpha))^n\mathbb{E}\Bigg[\frac{\varphi\big((r + U_n b^{n-1} + 1)b^{-n}\big) - \varphi\big((r + U_n b^{n-1})b^{-n}\big)}{b^{-n}}\Bigg] \\ &= \frac{(\mathrm{sgn}(\alpha)b)^n}{b}\sum_{k=0}^{b-1}\Big(\varphi\big((r+1)b^{-n} + k/b\big) - \varphi\big(rb^{-n} + k/b\big)\Big).\end{aligned} \tag{2.5}$$

If $n = 1$, then r must be zero, and the sum in (2.5) is a telescopic sum with value $\varphi(1) - \varphi(0) = 0$. Now consider the case $n \ge 2$. Then, for all $x \in \mathbb{R}$, $i = \sqrt{-1}$, and $\mathfrak{Re}$ denoting the real part of a complex number,

$$\sum_{k=0}^{b-1}\varphi(x + k/b) = \mathfrak{Re}\Big((\rho - i\nu)\sum_{k=0}^{b-1}e^{2\pi i(x+k/b)}\Big) = \mathfrak{Re}\Big((\rho - i\nu)e^{2\pi ix}\cdot\frac{e^{2\pi ib/b} - 1}{e^{2\pi i/b} - 1}\Big) = 0.$$

Therefore, the sum in (2.5) vanishes. □

Lemma 2.3. *With δ_x denoting the Dirac measure in $x \in \mathbb{R}$ and λ denoting the Lebesgue measure on $[0, 1]$, we have $\mathbb{P}$-a.s., $\frac{1}{n}\sum_{k=1}^n \delta_{b^{-k}R_k} \to \lambda$ weakly as $n \uparrow \infty$.*

Proof. Without loss of generality, we can extend the sequence $\{U_i\}_{i\in\mathbb{N}}$ to a two-sided sequence $\{U_i\}_{i\in\mathbb{Z}}$ of i.i.d. random variables with a uniform distribution on $\{0,\ldots,b-1\}$. Then we define $X_n := \sum_{j=1}^\infty U_{n+1-j}b^{-j} = \sum_{j=0}^\infty U_{n-j}b^{-(j+1)}$ for $n\in\mathbb{Z}$. Each X_n is uniformly distributed on $[0,1]$, i.e., has law λ. Moreover, in comparison with X_n, the random variable X_{n+1} is obtained by shifting the sequence $\{U_i\}_{i\in\mathbb{Z}}$ one step to the right. It is well-known that the dynamical system corresponding to such a two-sided Bernoulli shift is mixing and hence ergodic (for a proof, see, e.g., Example 20.26 in Klenke, 2014). By Birkhoff's ergodic theorem, we thus have $\frac{1}{n}\sum_{k=1}^n f(X_k)\to\int_0^1 f\,d\lambda$ $\mathbb{P}$-a.s. for each bounded Borel-measurable function on $[0,1]$. Since $|b^{-n}R_n - X_n|\le b^{-n}$, we hence obtain $\frac{1}{n}\sum_{k=1}^n f(b^{-k}R_k)\to\int_0^1 f\,d\lambda$ $\mathbb{P}$-a.s. for each (uniformly) continuous function on $[0,1]$. Since $C[0,1]$ is separable, the result follows. □

Proof of Theorem 1.3 for $t=1$. Let $\langle Z\rangle_n := \sum_{k=1}^n \mathbb{E}[Y_k^2|\mathscr{F}_{k-1}]$ be the predictable quadratic variation of the martingale $\{Z_n\}_{n\in\mathbb{N}_0}$. We define $\psi_n(x) := (\varphi(x+b^{-n})-\varphi(x))/b^{-n}$. Then $\psi_n(x)\to\varphi'(x)$ uniformly in x. By arguing as in (2.5), we see that $\mathbb{E}[Y_k^2|\mathscr{F}_{k-1}] = \frac{1}{b}\sum_{\ell=0}^{b-1}\big(\psi_k(b^{-k}R_{k-1}+\ell/b)\big)^2$. We therefore conclude from Lemma 2.3 that

$$\frac{1}{n}\langle Z\rangle_n = \frac{1}{b}\sum_{\ell=0}^{b-1}\sum_{k=1}^n\big(\psi_k(b^{-k}R_{k-1}+\ell/b)\big)^2 \longrightarrow \int_0^1\big(\varphi'(t)\big)^2\,dt = 2\pi^2(\nu^2+\rho^2) =: \sigma^2.$$

Analogously, one sees easily that $s_n^2 := \mathbb{E}[\langle Z\rangle_n]$ satisfies $\frac{1}{n}s_n^2\to\sigma^2$. Since the increments Y_k are uniformly bounded, the Lindeberg condition,

$$\frac{1}{n}\sum_{k=1}^n \mathbb{E}\big[Y_k^2\mathbb{1}_{\{Y_k^2\ge\varepsilon n\}}\big|\mathscr{F}_{k-1}\big]\longrightarrow 0 \qquad \mathbb{P}\text{-a.s. for all } \varepsilon>0,$$

is clearly satisfied. Therefore, the martingale central limit theorem in the form of Durrett (2005, (7.4) in Chapter 7) yields that the laws of $\frac{1}{\sqrt{n}}Z_n$ converge weakly to $N(0,\sigma^2)$. Lemma 2.1 hence gives

$$V_n \longrightarrow 2\sqrt{\frac{\pi(\nu^2+\rho^2)}{\log b}}. \quad \square$$

Finally, we show how the preceding results can be extended to the case $0\le t<1$. Writing Z_n for $\sum_{k=1}^n Y_k$, the Φ-variation over the interval $[0,t]$ is equal to

$$\begin{aligned}V_{n,t} &:= \sum_{k=0}^{b^n-1}\Phi\big(\big|f((k+1)b^{-n})-f(kb^{-n})\big|\big)\mathbb{1}_{[0,t]}(kb^{-n}) = b^n\mathbb{E}\Big[\Phi\big(\big|f((R_n+1)b^{-n})-f(R_nb^{-n})\big|\big)\mathbb{1}_{\{b^{-n}R_n\le t\}}\Big]\\ &= b^n\mathbb{E}\Big[\Phi\Big(b^{-n}\Big|\sum_{m=1}^n Y_m\Big|\Big)\mathbb{1}_{\{b^{-n}R_n\le t\}}\Big] = \mathbb{E}\Big[\frac{|Z_n|}{\sqrt{n\log b-\log|Z_n|}}\mathbb{1}_{\{|Z_n|>0\}}\mathbb{1}_{\{b^{-n}R_n\le t\}}\Big].\end{aligned}$$

Let $\delta>0$ be given and pick $m\in\mathbb{N}$ such that $b^{-m}\le\delta$. Clearly, $\{b^{-n}R_n\le t\}\subset\{b^{-n}R_{m,n}\le t\}$, where $R_{m,n}:=R_n-R_{n-m}=\sum_{k=n-m+1}^n U_kb^{k-1}$. In addition, we argue as in the proof of Lemma 2.1 and take $\beta\in(0,\log b)$ and $n_0\in\mathbb{N}$ such that $n\beta<n\log b-\log(Cn)$ for all $n\ge n_0$ and (2.4) holds. Therefore, for $n\ge m\vee n_0$.

$$V_{n,t}\le\frac{1}{\sqrt{n\beta}}\mathbb{E}\big[|Z_n|\mathbb{1}_{\{b^{-n}R_{m,n}\le t\}}\big]\le\frac{1}{\sqrt{n\beta}}\mathbb{E}\big[|Z_{n-m}|\mathbb{1}_{\{b^{-n}R_{m,n}\le t\}}\big]+\frac{1}{\sqrt{n\beta}}\mathbb{E}\Big[\Big|\sum_{k=n-m+1}^n Y_k\Big|\Big].$$

Clearly, the rightmost term converges to zero as $n\uparrow\infty$. Moreover, Z_{n-m} and $R_{m,n}$ are independent, and so

$$\limsup_{n\uparrow\infty}V_{n,t}\le\sqrt{\frac{2\sigma^2}{\pi\beta}}\limsup_{n\uparrow\infty}\mathbb{P}[b^{-n}R_{m,n}\le t]\le\sqrt{\frac{2\sigma^2}{\pi\beta}}\limsup_{n\uparrow\infty}\mathbb{P}[b^{-n}R_n\le t+\delta]=\sqrt{\frac{2\sigma^2}{\pi\beta}}(t+\delta),$$

where the second inequality follows from the fact that $b^{-n}R_{m,n}\ge b^{-n}R_n-\delta$ for $n>m$. Sending $\beta\uparrow\log b$ and $\delta\downarrow 0$ gives the desired upper bound.

To get a corresponding lower bound, we choose $\delta>0$ and m as in the upper bound. In addition, we choose $\varepsilon>0$. For $n>m\vee 1/\varepsilon^2$, we then get as in the proof of Lemma 2.1,

$$V_{n,t}\ge\mathbb{E}\Big[\frac{|Z_n|}{\sqrt{n\log b}}\mathbb{1}_{\{|\frac{1}{\sqrt{n}}Z_n|\ge\varepsilon\}}\mathbb{1}_{\{b^{-n}R_n\le t\}}\Big]\ge\mathbb{E}\Big[\frac{|Z_n|}{\sqrt{n\log b}}\mathbb{1}_{\{|\frac{1}{\sqrt{n}}Z_n|\ge\varepsilon\}}\mathbb{1}_{\{b^{-n}R_{m,n}\le t-\delta\}}\Big].$$

Now let C be a uniform upper bound for $|Y_k|$ and choose n_1 such that $mC\le\varepsilon\sqrt{n_1}$. Then, for $n\ge n_1\vee m\vee 1/\varepsilon^2$,

$$V_{n,t}\ge\mathbb{E}\Big[\frac{|Z_{n-m}|}{\sqrt{n\log b}}\mathbb{1}_{\{|\frac{1}{\sqrt{n}}Z_{n-m}|\ge 2\varepsilon\}}\mathbb{1}_{\{b^{-n}R_{m,n}\le t-\delta\}}\Big]-\frac{1}{\sqrt{n\log b}}\mathbb{E}\Big[\Big|\sum_{k=n-m+1}^n Y_k\Big|\Big].$$

Again, the second expectation on the right converges to zero. Using as before the independence of Z_{n-m} and $R_{m,n}$ now easily gives the desired lower bound. This concludes the proof of Theorems 1.2 and 1.3 for $0\le t<1$.

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