



A Limit Theorem for Bernoulli Convolutions and the Φ -Variation of Functions in the Takagi Class

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Abstract

We consider a probabilistic approach to compute the Wiener–Young Φ -variation of fractal functions in the Takagi class. Here, the Φ -variation is understood as a generalization of the quadratic variation or, more generally, the p th variation of a trajectory computed along the sequence of dyadic partitions of the unit interval. The functions Φ we consider form a very wide class of functions that are regularly varying at zero. Moreover, for each such function Φ , our results provide in a straightforward manner a large and tractable class of functions that have nontrivial and linear Φ -variation. As a corollary, we also construct stochastic processes whose sample paths have nontrivial, deterministic, and linear Φ -variation for each function Φ from our class. The proof of our main result relies on a limit theorem for certain sums of Bernoulli random variables that converge to an infinite Bernoulli convolution.

Keywords Wiener–Young Φ -variation · Takagi class · Pathwise Itô calculus · Infinite Bernoulli convolution · Central limit theorem · Stochastic process with prescribed Φ -variation

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1 Introduction

Probabilistic models of continuous random trajectories typically rely on solutions to stochastic differential equations or, more generally, on continuous semimartingales. The huge success of those models is in no small part due to the fact that they can be analyzed by means of Itô calculus. Itô calculus is however not restricted to the sample paths of a semimartingale: Föllmer [8] showed that every continuous function that admits a continuous quadratic variation along a fixed refining sequence of partitions can be used as an integrator for Itô integration and that a corresponding Itô formula holds. As shown in [10,20,24], this includes many functions previously studied in fractal analysis and geometry.

Cont and Perkowski [4] recently extended Föllmer's pathwise Itô formula to integrators with higher than quadratic variation, the so-called p th variation. While quadratic variation is based on the squared increments of a function, p th variation is based on the p th power of the increments, where $p \geq 1$. In the realm of stochastic processes, the sample paths of fractional Brownian motion with Hurst index $1/p$ constitute the best-known class of such trajectories. At the same time, recent applications, such as the theory of rough volatility models based on the seminal paper [11] by Gatheral et al., have intensified the interest in creating new models for "rough" phenomena beyond the case of quadratic variation. Also in this strand of literature, the focus has so far been on models with finite and nontrivial p th variation.

From a mathematical point of view, however, there is no reason why attention should be restricted to variation based on power functions. A natural and much broader concept is given by the notion of Φ -variation in the Wiener–Young sense. Here, for a function $\Phi : [0, \infty) \rightarrow [0, \infty)$, the Φ -variation of a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ along the dyadic partitions is given by

$$\langle f \rangle_t^\Phi := \lim_{n \uparrow \infty} \sum_{k=0}^{\lfloor t2^n \rfloor} \Phi(|f((k+1)2^{-n}) - f(k2^{-n})|), \quad 0 \leq t \leq 1, \quad (1.1)$$

provided that the limit exists for all t . Yet, most authors still focus exclusively on the case of power variation. Notable exceptions are the paper [18] by Marcus and Rosen, in which the Φ -variation of Gaussian processes with stationary increments and local times of symmetric Lévy processes is obtained, and Kôno [16]. Note that the Φ -variation along a fixed refining sequence of partitions as defined in (1.1) typically differs from the Φ -variation defined as a supremum taken over *all* finite partitions. For Gaussian processes, the latter concept was studied, e.g., by Taylor [27] for $\Phi(x) = x^2$ and by Kawada and Kôno [15] for more general functions Φ .

In this paper, we use a probabilistic approach to explore a class of fractal trajectories f that admit the Φ -variation $\langle f \rangle_t^\Phi = t$ for functions Φ of the form

$$\Phi(x) := \begin{cases} 0 & \text{if } x = 0, \\ x^p g(-p \log_2 x)^{-p/2} & \text{if } 0 < x < 1, \end{cases} \quad (1.2)$$

where $p \in [1, \infty)$ and $g : [0, \infty) \rightarrow (0, \infty)$ is any regularly varying function.¹ In doing so, we address several high-level issues. First, we illustrate that finite and nontrivial p th power variation, which has so far been the default paradigm for characterizing the roughness of trajectories, appears to be the exception rather than the rule. In particular, one should expect Φ -variation rather than power variation when studying continuous trajectories that do not arise as sample paths of a semimartingale. Second, for every function Φ of the form (1.2), we construct explicit examples of trajectories with finite linear Φ -variation along the dyadic partitions. These examples can be used in further analyses involving “rough” trajectories. Moreover, the idea underlying our construction of functions with prescribed Φ -variation can probably be extended to more flexible models and into the realm of stochastic processes.

The class of functions f , we consider here belong to the so-called Takagi class, which was introduced by Hata and Yamaguti [14] and motivated by Takagi’s [26] celebrated example of a continuous nowhere differentiable function. This class was chosen due to its richness and tractability and also because it is closely related to the Wiener–Lévy construction of Brownian motion. The Takagi class consists of all functions $f : [0, 1] \rightarrow \mathbb{R}$ that admit an expansion of the form

$$f(t) := \sum_{m=0}^{\infty} \alpha_m \varphi(2^m t), \quad t \in [0, 1], \tag{1.3}$$

where $\varphi(t) = \min_{z \in \mathbb{Z}} |z - t|$ is the tent map and $\{\alpha_m\}_{m \in \mathbb{N}_0}$ is a sequence of real numbers for which the series $\sum_{m=0}^{\infty} \alpha_m$ converges absolutely. The special choice $\alpha_m = a^m$ for some $a \in (-1, 1)$ yields the class of so-called Takagi–Landsberg functions, which, for $1/2 < |a| < 1$ and $p = -\log_{|a|} 2$, have finite nontrivial and linear p th variation as shown in [21]. For $|a| < 1/2$, it is well-known that f is of finite total variation. The borderline case $a = 1/2$ corresponds to the classical Takagi function, which is nowhere differentiable and hence not of finite total variation, even though $p = -\log_{1/2} 2 = 1$. Instead, for $|a| = 1/2$, the function f has linear Φ -variation, $\langle f \rangle_t^\Phi = t$, for $\Phi(x) = x \sqrt{\pi / (-2 \log_2 x)}$, as shown by the authors in [13]. Note that this function Φ is a special case of (1.2). Our method also extends to an even richer class of functions for which each scaled phase of the tent map in (1.3) is multiplied with an arbitrary sign [see (3.7) for details]. This latter class can be fitted to empirical time series and gives also rise to a class of stochastic processes if the signs are chosen randomly.

Our approach to the Φ -variation of functions f in the Takagi class depends on the parameter p in (1.2). For $p = 1$, our argument extends the approach via the central limit theorem for the Rademacher functions in the Faber–Schauder development of f as presented in [13]. For $p > 1$, however, a new limit theorem is needed, which might be of independent interest. If $\{\beta_m\}_{m \in \mathbb{N}_0}$ is a sequence of real numbers and Y_1, Y_2, \dots

¹ Recall that a measurable and strictly positive function g , which is defined on some interval $[a, \infty)$, is called regularly varying (at infinity) with index $\varrho \in \mathbb{R}$ if $g(\lambda x)/g(x) \rightarrow \lambda^\varrho$ as $x \uparrow \infty$ for all $\lambda > 0$. If $\varrho = 0$, the function g is also called slowly varying. Throughout this paper, we will always assume that all regularly varying functions are defined on $[0, \infty)$. This assumption can be made without loss of generality, as one can always consider the function $x \mapsto g(a \vee x)$.

is an i.i.d. sequence of symmetric Bernoulli random variables, we investigate the limit of the “convolutions”

$$\frac{1}{s_n} \sum_{m=1}^n \beta_{n-m} Y_m \quad \text{where} \quad s_n^2 := \sum_{m=0}^{n-1} \beta_m^2.$$

We show that if there exist $q > 0$, $b > 1$, and a slowly varying function ℓ such that

$$\lim_{n \rightarrow \infty} \frac{s_n^2}{2^{2qn} \ell(b^n)} = 1, \quad (1.4)$$

then

$$\frac{1}{s_n} \sum_{m=1}^n \beta_{n-m} Y_m \longrightarrow \sqrt{1 - 2^{-2q}} \sum_{m=1}^{\infty} 2^{-qm} Y_m \quad \text{in } L^\infty. \quad (1.5)$$

Note that the law of the limiting random variable Z is a scaled version of the infinite Bernoulli convolution with parameter 2^{-q} . In Sect. 2, we will present an extended version of this result in which the Y_m are neither required to be independent nor identically distributed. This approach to the Φ -variation via the condition (1.4) and the limiting result (1.5) is to some extent motivated by Gladyshev’s theorem [12] (see also the book [19] by Marcus and Rosen for a comprehensive survey). It extends previous work in [13,21,25].

The paper is structured as follows. In Sect. 2, we derive the convergence result (1.5) and the central limit theorem that will be needed for our results on the Φ -variation of functions in the Takagi class. Our main result on this topic, Theorem 3.3, is stated in Sect. 3. The proof of Theorem 3.3 is given in Sect. 4.

2 A Limit Theorem for Bernoulli-Type Convolutions

For some $p \in [1, \infty]$, let Y_1, Y_2, \dots be a sequence of random variables in $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space. We assume for simplicity that the L^p -norms of Y_m are uniformly bounded. Let furthermore $\{\beta_m\}_{m \in \mathbb{N}_0}$ be a sequence of real numbers and consider the random variables

$$Z_n := \sum_{m=1}^n \beta_{n-m} Y_m, \quad n \in \mathbb{N}.$$

We are first concerned with the limit of

$$\frac{1}{s_n} Z_n \quad \text{where} \quad s_n^2 := \sum_{m=0}^{n-1} \beta_m^2 \quad (2.1)$$

as $n \uparrow \infty$. Our corresponding result uses the concept of a slowly varying function, which was recalled in the footnote on page 3.

Theorem 2.1 *Suppose that $\beta_m \geq 0$ for all m and that there exist $q > 0$, $b > 1$, and a slowly varying function ℓ such that*

$$\lim_{n \rightarrow \infty} \frac{s_n^2}{2^{2qn} \ell(b^n)} = 1. \tag{2.2}$$

Then,

$$\frac{1}{s_n} Z_n \longrightarrow Z := \sqrt{2^{2q} - 1} \sum_{m=1}^{\infty} 2^{-qm} Y_m \text{ in } L^p.$$

Proof of Theorem 2.1 For $\delta \in (0, \delta_0]$, denote

$$\lambda := \sqrt{2^{2q} - 1}, \quad \lambda_{\delta}^+ := \sqrt{2^{2q} \frac{(1 + \delta)^2}{1 - \delta} - 1}, \quad \text{and} \quad \lambda_{\delta}^- := \sqrt{2^{2q} \frac{(1 - \delta)^2}{1 + \delta} - 1},$$

where $\delta_0 \in (0, \min\{1, q\})$ is chosen such that the arguments of the square roots are always positive. Thus,

$$2^{2q}(1 - \delta)^2 - (1 + \delta) > 0 \text{ for all } \delta \in (0, \delta_0]. \tag{2.3}$$

Let $\varepsilon > 0$ be given. We first choose $K \in \mathbb{N}$ such that

$$\lambda \sum_{m=K+1}^{\infty} 2^{-(q-\delta_0)m} < \varepsilon. \tag{2.4}$$

Then we choose $\delta_1 \in (0, \delta_0]$ such that

$$1 - 2^{-\delta m} < \varepsilon \text{ for } m = 0, \dots, K \text{ and all } \delta \in (0, \delta_1]. \tag{2.5}$$

Finally, we choose $\delta \in (0, \delta_1]$ such that

$$\left| \frac{\lambda_{\delta}^{\pm}}{\sqrt{1 \mp \delta}} - \lambda \right| < \varepsilon. \tag{2.6}$$

Given this δ , we use (2.2) to pick $n_0 \in \mathbb{N}$ such that

$$\ell(b^n)2^{2qn}(1 - \delta) < s_n^2 < \ell(b^n)2^{2qn}(1 + \delta) \text{ for all } n > n_0. \tag{2.7}$$

By applying (2.7) to both s_{m+1}^2 and s_m^2 , subtracting the results, and using the definition of s_m from (2.1), we thus get that for $m > n_0$,

$$2^{2q(m+1)}((1-\delta)\ell(b^{m+1}) - (1+\delta)\ell(b^m)2^{-2q}) < \beta_m^2 < 2^{2q(m+1)}((1+\delta)\ell(b^{m+1}) - (1-\delta)\ell(b^m)2^{-2q}). \quad (2.8)$$

Since ℓ is slowly varying, we can choose $n_1 \geq n_0$ such that

$$1 - \delta < \frac{\ell(b \cdot b^m)}{\ell(b^m)} < 1 + \delta \quad \text{for all } m \geq n_1.$$

Combining this with (2.3), we see that the left-hand side of (2.8) is strictly positive for $m \geq n_1$. Hence, we may take square roots to get

$$\begin{aligned} 2^{q(m+1)}\sqrt{(1-\delta)\ell(b^{m+1}) - (1+\delta)\ell(b^m)2^{-2q}} &< \beta_m \\ &< 2^{q(m+1)}\sqrt{(1+\delta)\ell(b^{m+1}) - (1-\delta)\ell(b^m)2^{-2q}} \end{aligned} \quad (2.9)$$

for all $m \geq n_1$. Moreover, with (2.7), we get that for $n > m \geq n_1$,

$$\begin{aligned} \frac{\beta_m}{s_n} &< \frac{2^{q(m+1)}\sqrt{(1+\delta)\ell(b^{m+1}) - (1-\delta)\ell(b^m)2^{-2q}}}{\sqrt{(1-\delta)2^{2qn}\ell(b^n)}} \\ &= 2^{q(m-n)}\sqrt{\frac{\ell(b^m)}{\ell(b^n)}\left(2^{2q}\frac{1+\delta}{1-\delta}\frac{\ell(b^{m+1})}{\ell(b^m)} - 1\right)} \\ &< 2^{q(m-n)}\sqrt{\frac{\ell(b^m)}{\ell(b^n)}}\lambda_\delta^+. \end{aligned}$$

Similarly, we conclude a corresponding lower bound and obtain that for $n > m \geq n_1$,

$$2^{-q(n-m)}\sqrt{\frac{\ell(b^m)}{\ell(b^n)}}\lambda_\delta^- < \frac{\beta_m}{s_n} < 2^{-q(n-m)}\sqrt{\frac{\ell(b^m)}{\ell(b^n)}}\lambda_\delta^+. \quad (2.10)$$

The Potter bounds (see, e.g., Theorem 1.5.6 in [3]) yield that there is $n_2 \geq n_1$ such that for $n \geq m \geq n_2$,

$$\begin{aligned} (1-\delta)2^{-2\delta(n-m)} &= (1-\delta)b^{-2\delta(n-m)\log_b 2} \leq \frac{\ell(b^n)}{\ell(b^m)} \leq (1+\delta)b^{2\delta(n-m)\log_b 2} \\ &= (1+\delta)2^{2\delta(n-m)}. \end{aligned}$$

With (2.10), we obtain that for $n \geq n_2 + m$,

$$2^{-(q+\delta)m}\frac{\lambda_\delta^-}{\sqrt{1+\delta}} < \frac{\beta_{n-m}}{s_n} < 2^{-(q-\delta)m}\frac{\lambda_\delta^+}{\sqrt{1-\delta}}. \quad (2.11)$$

Note that

$$Z^{(n)} := \lambda \sum_{m=1}^n 2^{-qm} Y_m \longrightarrow Z \text{ in } L^p.$$

By Minkowski’s inequality and with $c := \sup_m \|Y_m\|_p < \infty$,

$$\begin{aligned} \left\| \frac{Z_n}{s_n} - Z^{(n)} \right\|_p &\leq \sum_{m=1}^n \left| \frac{\beta_{n-m}}{s_n} - \lambda 2^{-qm} \right| \|Y_m\|_p \leq c \sum_{m=1}^{n-n_2} \left| \frac{\beta_{n-m}}{s_n} - \lambda 2^{-qm} \right| \\ &\quad + c \sum_{m=n-n_2+1}^n \left| \frac{\beta_{n-m}}{s_n} - \lambda 2^{-qm} \right|. \end{aligned} \tag{2.12}$$

To deal with the first sum on the right-hand side, we use (2.6) and (2.11) to get that for $m \leq n - n_2$,

$$\begin{aligned} \frac{\beta_{n-m}}{s_n} - \lambda 2^{-qm} &\leq 2^{-(q-\delta)m} \left(\frac{\lambda_\delta^+}{\sqrt{1-\delta}} - \lambda \right) + \lambda (2^{-(q-\delta)m} - 2^{-qm}) \\ &\leq \varepsilon 2^{-(q-\delta)m} + \lambda 2^{-(q-\delta)m} (1 - 2^{-\delta m}). \end{aligned}$$

In the same way, we get the lower bound

$$-\varepsilon 2^{-(q+\delta)m} - \lambda 2^{-qm} (1 - 2^{-\delta m}) \leq \frac{\beta_{n-m}}{s_n} - \lambda 2^{-qm}.$$

Using (2.5) and (2.4), we thus get the following estimate for the first sum on the right-hand side of (2.12)

$$\begin{aligned} \sum_{m=1}^{n-n_2} \left| \frac{\beta_{n-m}}{s_n} - \lambda 2^{-qm} \right| &\leq \varepsilon \sum_{m=1}^{n-n_2} 2^{-(q-\delta)m} + \lambda \sum_{m=1}^K 2^{-(q-\delta)m} \varepsilon + \lambda \sum_{m=K+1}^\infty 2^{-(q-\delta)m} \\ &\leq \varepsilon c', \end{aligned}$$

where $c' := 2 \sum_{m=0}^\infty 2^{-(q-\delta_0)m} + 1$. For the right-most sum in (2.12), we have

$$\begin{aligned} \sum_{m=n-n_2+1}^n \left| \frac{\beta_{n-m}}{s_n} - \lambda 2^{-qm} \right| &\leq \sum_{m=n-n_2+1}^n \frac{\beta_{n-m}}{s_n} + \lambda \sum_{m=n-n_2+1}^n 2^{-qm} \\ &\leq \frac{n_2}{s_n} \max_{i=0, \dots, n_2-1} \beta_i + \lambda 2^{-qn} \frac{2^{qn_2} - 1}{2^q - 1}. \end{aligned} \tag{2.13}$$

We will show below that our assumptions imply that $s_n \rightarrow \infty$. Therefore, the right-hand side of (2.13) can be made smaller than ε by choosing n sufficiently large. Thus,

we see that the L^p -distance between Z_n/s_n and $Z^{(n)}$ is smaller than $c(c' + 1)\varepsilon$ if n is sufficiently large.

Finally, we argue that $s_n \rightarrow \infty$ if (2.2) holds and $q > 0$. To this end, let $g(x) := x^{2q}\ell(x)$, so that the denominator in (2.2) is equal to $g(2^n)$. Proposition 1.3.6 in [3] states that $g(x) \rightarrow \infty$, which in view of (2.2) yields our claim. \square

Now, we briefly discuss the case where $q = 0$ in (2.2). In that case, we are typically in the regime of the central limit theorem. This is suggested, for instance, by [28], where a central limit theorem for reversible Markov chains was obtained under the condition that the variance, σ_n^2 , satisfies $\sigma_n^2 = n\ell(n)$ for a slowly varying function ℓ . Since $g(x) := x\ell(x)$ is regularly varying with index $\varrho = 1$, we see that, in our situation, this condition is a special case of condition (3.4) with $q = 0$. In the following proposition, we formulate implications between stronger and weaker versions of condition (2.2) with $q = 0$.

Now, we are going to formulate a simple central limit theorem in the form needed in the next section. We believe that this result is well-known but were unable to find an exact reference to the literature. Due to the large number of existing variants of the central limit theorem, we confine ourselves here to the basic case that will actually be needed in Sect. 4.

Lemma 2.2 *Suppose that Y_1, Y_2, \dots is an i.i.d. sequence of symmetric $\{-1, +1\}$ -valued Bernoulli random variables such that $s_{n-1}^2/s_n^2 \rightarrow 1$ as $n \uparrow \infty$. Then, for every $p \in [1, \infty)$, the laws of $\frac{1}{s_n}Z_n$ converge in the L^p -Wasserstein metric to the standard normal distribution.*

Proof As $s_{n-1}^2/s_n^2 \rightarrow 1$, it then follows that

$$\lim_{n \uparrow \infty} \frac{\beta_n^2}{s_n^2} = \lim_{n \uparrow \infty} \left(1 - \frac{s_{n-1}^2}{s_n^2} \right) = 0.$$

It is hence easy to see that the sequence $\{Z_n\}_{n \in \mathbb{N}_0}$ satisfies Lindeberg’s condition, which yields that $\frac{1}{s_n}Z_n$ converges in law to $N(0, 1)$. Thus, to conclude convergence in the L^p -Wasserstein metric, it is sufficient to show that $\mathbb{E}[|s_n^{-1}Z_n|^q]$ is uniformly bounded in n for some $q > p$; this follows, e.g., from Lemma 5.61 and Corollary A.50 in [9]. We actually show a stronger condition, namely that $\mathbb{E}[e^{\lambda Z_n/s_n}]$ is uniformly bounded for all $\lambda \in \mathbb{R}$. Since $e^{|\lambda x|} \leq e^x + e^{-x}$, we can drop the absolute value in the exponent. Then,

$$\log \mathbb{E}[e^{\lambda Z_n/s_n}] = \log \prod_{k=0}^n \cosh(\lambda \beta_k/s_n) = \sum_{k=0}^n \log \cosh(\lambda \beta_k/s_n) \leq \sum_{k=0}^n \frac{\lambda^2 \beta_k^2}{2s_n^2} = \frac{\lambda^2}{2}.$$

\square

3 Φ -variation of Functions in the Takagi Class

In this section, we will apply the limit theorems from Sect. 2 to the problem of computing the Φ -variation of functions in the Takagi class. To this end, let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a function with $\Phi(0) = 0$. The Φ -variation of a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ along the sequence of dyadic partitions is defined as

$$\langle f \rangle_t^\Phi := \lim_{n \uparrow \infty} \sum_{k=0}^{\lfloor t2^n \rfloor} \Phi(|f((k+1)2^{-n}) - f(k2^{-n})|), \quad 0 \leq t \leq 1,$$

provided that the limit exists for all $t \in [0, 1]$. Due to the uniform continuity of f , we may, without loss of generality, restrict Φ to the domain $[0, 1)$. For $\Phi(x) = x$ and $t = 1$, we obtain the total variation of f . For $\Phi(x) = x^2$, we obtain the usual quadratic variation, and for $\Phi(x) = x^p$ with $p \geq 1$ the p th variation. Here, we are interested in functions for which the correct Φ may not be of the form $\Phi(x) = x^p$ but rather has a more complex structure.

Let $\varphi(t) = \min_{z \in \mathbb{Z}} |z - t|$ denote the tent map. The Takagi class, as introduced by Hata and Yamaguti [14], consists of all functions $f : [0, 1] \rightarrow \mathbb{R}$ that admit an expansion of the form:

$$f(t) := \sum_{m=0}^{\infty} \alpha_m \varphi(2^m t), \quad t \in [0, 1], \tag{3.1}$$

where $\{\alpha_m\}_{m \in \mathbb{N}_0}$ is a sequence of real numbers for which the series $\sum_{m=0}^{\infty} \alpha_m$ converges absolutely. Clearly, the series on the right-hand side of (3.1) converges uniformly in $t \in [0, 1]$, so that f is indeed a well-defined continuous function. Typically, the functions in the Takagi class have a fractal structure. For instance, the choice $\alpha_m = 2^{-m}$ corresponds to the classical, nowhere differentiable Takagi function, which was originally introduced in [26] but rediscovered many times. More generally, the choice $\alpha_m = a^m$ for some $a \in (-1, 1)$ yields the class of so-called Takagi–Landsberg functions. Analytical properties of the functions in the Takagi class, such as differentiability, are discussed in [17].

In our first result, we take f as in (3.1) as given and relate the asymptotic behavior of

$$V_n^{(p)} := \sum_{k=0}^{2^n-1} |f((k+1)2^{-n}) - f(k2^{-n})|^p, \quad p \geq 1, \quad n \in \mathbb{N},$$

to the one of

$$s_n^2 := \sum_{m=0}^{n-1} \alpha_m^2 4^m, \quad n \in \mathbb{N}. \tag{3.2}$$

Note that $V_n^{(1)}$ converges to the total variation of f , defined as usual through taking the supremum over *all* possible partitions of $[0, 1]$ (see, e.g., Theorem 2 in §5 of Chapter VIII in [22]). The following proposition provides a priori bounds on the behavior of $V_n^{(p)}$, based on the range of possible growth rates of s_n .

Proposition 3.1 *Let*

$$q_* := \liminf_{n \uparrow \infty} \frac{1}{n} \log_2 s_n \quad \text{and} \quad q^* := \limsup_{n \uparrow \infty} \frac{1}{n} \log_2 s_n.$$

Then, for $p \geq 1$, we have $\liminf_n V_n^{(p)} = 0$ if $p(1 - q_) > 1$ and $\limsup_n V_n^{(p)} = \infty$ if $p(1 - q^*) < 1$.*

Example 3.2 Faber [7] considered the function f with

$$\alpha_m = \begin{cases} \frac{1}{10^n} & \text{if } m = n! \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

and showed that it is nowhere differentiable. It is easy to see that this choice leads to $q^* \geq 1$ and $q_* \leq 0$, which implies that $\liminf_n V_n^{(p)} = 0$ and $\limsup_n V_n^{(p)} = \infty$ for all $p \geq 1$.

A more interesting case is the situation in which f is such that $q := \lim_n \frac{1}{n} \log_2 s_n$ exists and satisfies $0 \leq q < 1$. Then, Proposition 3.1 implies that $p := 1/(1 - q)$ is the critical exponent for the power variation of f . The standard paradigm in the literature is to expect that f then admits a finite p th variation. This is true for the typical sample paths of a continuous semimartingale with $p = 2$ and for the trajectories of fractional Brownian motion with Hurst exponent $H = 1/p$. As shown in [21, 25], it is also true for the Takagi–Landsberg functions. In the sequel, we shall in particular investigate the situation in which $\lim_n V_n^{(p)}$ exists but is either zero or infinity. In this case, p th power variation is clearly no longer sufficient for characterizing the exact “roughness” of f .

To state our first main result, we recall that Corollary 1.4.2 in [3] allows us to assume without loss of generality that all regularly varying functions are bounded on compact intervals (see the footnote on page 2 for the definition of a regularly varying function). If g is regularly varying and $q \in [0, 1)$, consider

$$\Phi_q(x) = \Phi_{q,g}(x) := x^{\frac{1}{1-q}} g \left(\frac{-\log_2 x}{1-q} \right)^{-\frac{1}{2(1-q)}} \quad (3.3)$$

for $0 < x < 1$. By Proposition 1.5.7 in [3], $y \mapsto g((\log_2 y)/(1 - q))$ is slowly varying, and so $\psi_q(y) := \Phi_q(1/y)$ is regularly varying with index $-1/(1 - q)$. We conclude that $\Phi_q(x) \rightarrow 0$ as $x \downarrow 0$. It therefore makes sense to extend Φ_q to the domain $[0, 1)$ by setting $\Phi_q(0) := 0$. Note also that Φ_q is *slowly varying at zero* with index $1/(1 - q)$ and that the functions arising in this manner form a very wide class of functions that are regularly varying at zero.

Theorem 3.3 For f as in (3.1), let s_n^2 be as in (3.2) and suppose that there exist $q \in [0, 1)$ and a regularly varying function g such that

$$\lim_{n \uparrow \infty} \frac{s_n^2}{2^{2qn} g(n)} = 1. \tag{3.4}$$

Let moreover $\{Y_m\}_{m \in \mathbb{N}}$ be an i.i.d. sequence of symmetric $\{-1, +1\}$ -valued Bernoulli random variables and Φ_q as in (3.3). Then, the following holds:

- (a) The function f is of bounded variation if and only if $\lim_n s_n^2 < \infty$ and, in this case, the total variation of f is equal to $\mathbb{E}[|\tilde{Z}|]$, where $\tilde{Z} := \sum_{m=0}^{\infty} \alpha_m 2^m Y_{m+1}$.
- (b) If $s_n^2 \rightarrow \infty$ and $q = 0$, then the Φ_0 -variation of f is given by

$$\langle f \rangle_t^{\Phi_0} = \sqrt{\frac{2}{\pi}} \cdot t. \tag{3.5}$$

- (c) If $q > 0$, then the Φ_q -variation of f is given by

$$\langle f \rangle_t^{\Phi_q} = \mathbb{E}[|Z|^{\frac{1}{1-q}}] \cdot t \quad \text{where} \quad Z := \sqrt{2^{2q} - 1} \sum_{m=1}^{\infty} 2^{-qm} Y_m. \tag{3.6}$$

Remark 3.4 The random variable Z in (3.6) can be represented as

$$Z = \sqrt{1 - 2^{-2q}} \sum_{m=0}^{\infty} 2^{-qm} Y_{m+1}$$

and hence has as its law a scaled version of the infinite Bernoulli convolution with parameter 2^{-q} . These laws exhibit some fascinating properties and have been studied in their own right for many decades; see, e.g., [23] and the references therein.

In the next two results, we consider again the situation of Proposition 3.1 and take a closer look at what happens to the power variation of a function f in the Takagi class satisfying (3.4).

Corollary 3.5 Suppose that (3.4) holds with $q \in [0, 1)$ and let $p := 1/(1 - q)$.

- (a) The function f has infinite r th variation for $1 \leq r < p$ and vanishing r th variation for $r > p$.
- (b) If $\lim_n g(n) = c$ for some number $c \in [0, \infty]$, then the p th variation of f satisfies $\langle f \rangle_t^{(p)} = c^{p/2} \langle f \rangle_t^{\Phi_q}$ for $t \in [0, 1]$.

If g is regularly varying with index $\varrho \in \mathbb{R}$, then $g(x) \rightarrow 0$ for $\varrho < 0$ and $g(x) \rightarrow \infty$ for $\varrho > 0$ according to Proposition B.1.9 (1) in [5]. If $\varrho = 0$, then g is slowly varying and may or may not converge to a number $c \in [0, \infty]$. In conjunction with Corollary 3.5, we thus get immediately the following corollary.

Corollary 3.6 Suppose that (3.4) holds with $q \in [0, 1)$ and a function g that is regularly varying with index $\varrho \in \mathbb{R}$. We let $p := 1/(1 - q)$.

- (a) If $q < 0$, then f has vanishing p th variation.
- (b) If $q > 0$, then f has infinite p th variation.

Example 3.7 Let us consider the classical Takagi function,

$$f(t) = \sum_{m=0}^{\infty} 2^{-m} \varphi(2^m t),$$

as discussed in the introduction. Then, $s_n^2 = n$ and so we can take $q = 0$ and $g(x) := x$. Corollary 3.6 thus immediately yields the well-known fact that f is of unbounded total variation. Moreover, we have $\Phi_0(t) = x/\sqrt{-\log_2 x}$, and so we recover here a special case of Theorem 1.2 in [13].

Allaart [1] introduced an extension of the Takagi class by multiplying each replicate of the tent map in the expansion (3.1) with an arbitrary sign. More precisely, he considered the class of functions of the form

$$f(t) := \sum_{m=0}^{\infty} \alpha_m \sigma_m(t) \varphi(2^m t), \quad t \in [0, 1], \tag{3.7}$$

where $\sigma_m : [0, 1] \rightarrow \{-1, +1\}$ is constant on each interval $[(k - 1)2^{-m}, k2^{-m})$ for $k = 1, \dots, 2^m$, and $\{\alpha_m\}_{m \in \mathbb{N}_0}$ is a sequence of real numbers for which the series $\sum_m \alpha_m$ converges absolutely. Then, f is well-defined and continuous due to the uniform convergence of the series in (3.7). Certain deterministic choices for σ_m yield fractal functions studied in other fields such as information theory; see [1,2] and the references therein. When the signs of σ_m are chosen in a random manner, the expansion (3.7) becomes closely related to the Wiener–Lévy–Ciesielski expansion of Brownian motion by means of the Faber–Schauder functions, and the functions (3.7) with fixed coefficients $\{\alpha_m\}_{m \in \mathbb{N}_0}$ but random signs form a non-Gaussian stochastic process with rough sample paths. More precisely, for each m , let $\{\xi_{m,k}\}_{k=1,\dots,2^m}$ be a sequence of $\{-1, +1\}$ -valued Bernoulli random variables and define

$$\sigma_m(t) := \sum_{k=1}^{2^m} \xi_{m,k} \mathbb{1}_{[(k-1)2^{-m}, k2^{-m})}(t) \quad \text{and} \quad X_t := \sum_{m=0}^{\infty} \alpha_m \sigma_m(t) \varphi(2^m t). \tag{3.8}$$

Then, the following corollary states that $\{X_t\}_{t \in [0,1]}$ is a stochastic process whose sample paths have the deterministic linear Φ_q -variation (3.6) if the sequence $\{\alpha_m\}_{m \in \mathbb{N}_0}$ satisfies the conditions on Theorem 3.3. An illustration of this stochastic process is provided in Fig. 1. For the choice $\alpha_m = a^m$ with $a \in (-1, 1)$, the p th variation of the functions (3.7) was studied in [21,24].

Corollary 3.8 *The statements of Theorem 3.3 and of the Corollaries 3.5 and 3.6 remain fully valid for the functions of the form (3.7).*

When the goal is to utilize Theorem 3.3 for the construction of functions with prescribed Φ_q -variation for a given function g , it can be inconvenient that the condition

(3.4) is formulated in terms of the sums $s_n^2 = \sum_{m=0}^{n-1} \alpha_m^2 4^m$ rather than in terms of the coefficients α_m themselves. To deal with this issue, we are going to formulate Proposition 3.10, which will also be used in the proofs of our main results.

Remark 3.9 Let us recall the following facts on slowly and regularly varying functions so as to put the statement of the following Proposition 3.10 into the context of Theorem 3.3.

- (a) If g is a regularly varying function and $b > 1$, then $\ell(x) := g(\log_b x)$ is slowly varying according to Proposition 1.5.7 (ii) in [3]. Moreover, our condition (3.4) implies that

$$\lim_{n \rightarrow \infty} \frac{s_n^2}{2^{2qn} \ell(b^n)} = 1.$$

- (b) Recalling that we assume all regularly (and hence slowly) varying functions to be locally bounded, and thus locally integrable, we may define

$$\ell(x) := \frac{1}{\log b} \int_1^x \frac{1}{t} L(t) dt = \int_0^{\log_b x} L(b^s) ds \tag{3.9}$$

for a given slowly varying function L . Then, according to Proposition 1.5.9 (a) in [3], the function ℓ is slowly varying and satisfies $\ell(x)/L(x) \rightarrow +\infty$ as $x \uparrow \infty$.

Proposition 3.10 *Let $\{\beta_m\}_{m \in \mathbb{N}_0}$ be a sequence of real numbers, $s_n^2 = \sum_{m=0}^{n-1} \beta_m^2$, and $b > 1$.*

- (a) *Suppose that $s_n \rightarrow \infty$ as $n \uparrow \infty$, let L be a slowly varying function, define ℓ as in (3.9), and consider the following two conditions:*
 - (i) $\beta_n^2/L(b^n) \rightarrow 1$ as $n \uparrow \infty$.
 - (ii) $s_n^2/\ell(b^n) \rightarrow 1$ as $n \uparrow \infty$.

Then, (i) \Rightarrow (ii).

- (b) *If ℓ is a slowly varying function and $q > 0$, the following are equivalent:*

- (i) $\beta_n^2/2^{2qn} \ell(b^n) \rightarrow 1$ as $n \uparrow \infty$.
- (ii) $s_n^2/2^{2qn} \ell(b^n) \rightarrow (2^{2q} - 1)^{-1}$ as $n \uparrow \infty$.

- (c) *Any of the conditions (i)–(iv) implies $s_{n-1}^2/s_n^2 \rightarrow 2^{2q}$ as $n \uparrow \infty$, where we take $q = 0$ in the context of (a).*

The proofs of Propositions 3.1, 3.10, Theorem 3.3, Corollaries 3.8, and 3.5, are given in Sect. 4. Theorem 3.3 and Proposition 3.10 yield immediately the following result, which provides a simple construction for a function with prescribed Φ_q -variation. When applying it to the stochastic process (3.8), it provides a straightforward way of constructing a stochastic process whose sample paths have deterministic and linear Φ_q -variation; see Fig. 1 for an illustration.

Corollary 3.11 *Let g be a regularly varying function, $q \in [0, 1)$, and Φ_q as in (3.3).*

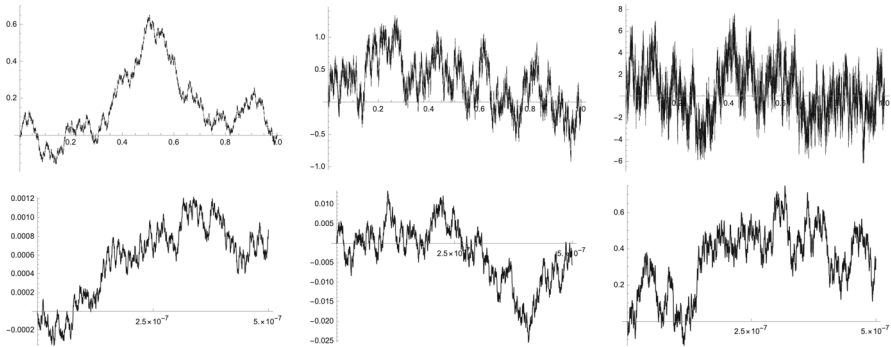


Fig. 1 Sample paths of the stochastic process $\{X_t\}_{t \in [0,1]}$ defined in (3.8) with $\alpha_m = 2^{-m(1-q)} \sqrt{(2^{2q} - 1)g(m)}$ for $q = 0.7$, $g(x) = (1+x)^\varrho$ with $\rho = -2$ (left), $\rho = 0$ (center), $\rho = 2$ (right), and signs $\{\xi_{m,k}\}_{m \in \mathbb{N}_0, k=1, \dots, 2^m}$ that are i.i.d. $\{-1, +1\}$ -valued Bernoulli random variables. For each ϱ , a new realization of $\{\xi_{m,k}\}$ was chosen. One can see from the three top panels that the parameter ϱ has a strong influence on the macroscopic fluctuations of the trajectories. The fact that all three functions share the same critical exponent $p = 1/(1 - q) = 10/3$ as defined in Proposition 3.1 suggest a common Hurst exponent of $H = 1/p = 0.3$ (see the discussion on page 8). One would thus expect the fluctuations of the three functions to be similar on a microscopic scale. This effect can indeed be observed in the lower panels below, which show the same three functions after zooming into the interval $[0, \varepsilon]$ for $\varepsilon = 5 \cdot 10^{-7}$

(a) For $q \in (0, 1)$, the functions f in (3.1) and (3.7) with coefficients

$$\alpha_m := 2^{-m(1-q)} \sqrt{(2^{2q} - 1)g(m)}$$

have linear Φ_q -variation (3.6).

(b) For $q = 0$, we assume in addition that $g(x) \rightarrow \infty$ as $x \uparrow \infty$ and that g is absolutely continuous with regularly varying derivative g' . Then, the functions f in (3.1) and (3.7) with coefficients

$$\alpha_m := 2^{-m} \sqrt{g'(m)}$$

have linear Φ_q -variation (3.5).

Proof (b) Since the function $L(x) := g'(\log_2 x)$ is slowly varying by Remark 3.9, Proposition 3.10 (a) yields that $s_n^2/\ell(2^n) \rightarrow 1$, where $\ell(x) = \int_0^{\log_2 x} L(2^t) dt = g(\log_2 x) - g(0)$. Hence, the conditions of Theorem 3.3 (b) are satisfied. Part (a) is now obvious. \square

4 Proofs

Proof of Proposition 3.10 (i) \Rightarrow (ii): For any given $\varepsilon > 0$, there exists n_0 such that for all $m > n_0$, we have $L(b^m)(1 - \varepsilon) < \beta_m^2 < L(b^m)(1 + \varepsilon)$. Hence, for all $v \geq n_0$ and $n > v$,

$$\sum_{m=0}^v \beta_m^2 + (1 - \varepsilon) \sum_{m=v+1}^{n-1} L(b^m) < \sum_{m=0}^{n-1} \beta_m^2 < \sum_{m=0}^v \beta_m^2 + (1 + \varepsilon) \sum_{m=v+1}^{n-1} L(b^m). \tag{4.1}$$

The Potter bounds (e.g., Theorem 1.5.6 in [3]) yield for every $\delta > 0$ some $n_1 \in \mathbb{N}$ such that

$$(1 - \varepsilon)b^{\delta(m-t)} \leq \frac{L(b^t)}{L(b^m)} \leq (1 + \varepsilon)b^{\delta(t-m)} \quad \text{for all } t, m \geq n_1.$$

We hence get that for $m \geq n_1$,

$$L(b^m)(1 - \varepsilon) \frac{1 - b^{-\delta}}{\delta \log b} \leq \int_m^{m+1} L(b^t) dt \leq L(b^m)(1 + \varepsilon) \frac{b^\delta - 1}{\delta \log b}.$$

Since $\pm(1 - b^{\mp\delta})/(\delta \log b) \rightarrow 1$ as $\delta \downarrow 0$, we can choose δ small enough such that for any $m \geq n_1$,

$$(1 - \varepsilon)^2 L(b^m) \leq \int_m^{m+1} L(b^t) dt \leq (1 + \varepsilon)^2 L(b^m).$$

Using this estimate in (4.1) and dividing both sides of the result by $\int_0^n L(b^t) dt = \ell(b^n)$ gives

$$\frac{\sum_{m=0}^{n_1-1} \beta_m^2}{\int_0^n L(b^t) dt} + \frac{(1 - \varepsilon)^3 \int_{n_1}^n L(b^t) dt}{\int_0^n L(b^t) dt} < \frac{s_n^2}{\ell(b^n)} < \frac{\sum_{m=0}^{n_1-1} \beta_m^2}{\int_0^n L(b^t) dt} + \frac{(1 + \varepsilon)^3 \int_{n_1}^n L(b^t) dt}{\int_0^n L(b^t) dt}. \tag{4.2}$$

Since $s_n \rightarrow \infty$, the inequalities in (4.2) imply that we must also have $\ell(b^n) = \int_0^n L(b^t) dt \rightarrow \infty$ as $n \uparrow \infty$. It follows that the left- and right-hand sides of (4.2) tend to $(1 - \varepsilon)^3$ and $(1 + \varepsilon)^3$ as $n \uparrow \infty$, respectively. Sending $\varepsilon \downarrow 0$ now yields (ii).

(iv) \Leftrightarrow (iii): It follows from the Stolz theorem and its converse given in Lemma 3.1 of [20] that the existence of one of the following limits entails the existence of the other one and that in this case both must be equal:

$$\lim_{n \uparrow \infty} \frac{s_n^2}{2^{2qn} \ell(b^n)} \quad \text{and} \quad \lim_{n \uparrow \infty} \frac{s_{n+1}^2 - s_n^2}{2^{2q(n+1)} \ell(b^{n+1}) - 2^{2qn} \ell(b^n)}.$$

Moreover, if it exists, the limit on the right-hand side is equal to

$$\lim_{n \uparrow \infty} \frac{\beta_n^2}{2^{2qn} \ell(b^n)} \cdot \frac{1}{(2^{2q} \frac{\ell(b \cdot b^n)}{\ell(b^n)} - 1)},$$

and here the second factor converges to $(2^{2q} - 1)^{-1}$.

(c): We have

$$\lim_{n \uparrow \infty} \frac{s_{n-1}^2}{s_n^2} = 2^{2q} \lim_{n \uparrow \infty} \frac{\ell(b^{n-1})}{\ell(b \cdot b^{n-1})} = 2^{2q}.$$

This concludes the proof. □

Now, we turn toward proving Theorem 3.3 (and also Corollary 3.8 along the way). Following [13,25], we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting an independent sequence U_1, U_2, \dots of symmetric $\{0, 1\}$ -valued Bernoulli random variables. Then, we define the stochastic process $R_m := \sum_{k=1}^m U_k 2^{k-1}$ and set

$$Y_m := \frac{\varphi((R_m + 1)2^{-m}) - \varphi(R_m 2^{-m})}{2^{-m}}, \quad \beta_m := 2^m \alpha_m, \quad \text{and} \quad s_n^2 := \sum_{m=0}^{n-1} \beta_m^2. \tag{4.3}$$

Proposition 3.2 (a) in [25] states that Y_1, Y_2, \dots is an i.i.d. sequence of symmetric $\{-1, +1\}$ -valued Bernoulli random variables. Moreover, the proof of Theorem 2.1 in [21] gives the same result in the context of Corollary 3.8, so that all subsequent observations are also valid for the functions of the form (3.7).

Note that R_m has a uniform distribution on $\{0, \dots, 2^m - 1\}$. Therefore, for $n \in \mathbb{N}$ such that all increments $|f((k + 1)2^{-n}) - f(k2^{-n})|$ are less than 1,

$$\begin{aligned} V_n &:= \sum_{k=0}^{2^n-1} \Phi(|f((k + 1)2^{-n}) - f(k2^{-n})|) \\ &= 2^n \mathbb{E} \left[\Phi(|f((R_n + 1)2^{-n}) - f(R_n 2^{-n})|) \right]. \end{aligned}$$

To analyze the expectation on the right, let the n th truncation of f be given by $f_n(t) = \sum_{m=0}^{n-1} \alpha_m \varphi(2^m t)$. With the notation introduced in (4.3), we get

$$\begin{aligned} f((R_n + 1)2^{-n}) - f(R_n 2^{-n}) &= f_n((R_n + 1)2^{-n}) - f_n(R_n 2^{-n}) \\ &= 2^{-n} \sum_{m=0}^{n-1} \beta_m \frac{\varphi((R_n + 1)2^{m-n}) - \varphi(R_n 2^{m-n})}{2^{m-n}} \\ &= 2^{-n} \sum_{m=0}^{n-1} \beta_m Y_{n-m}, \end{aligned}$$

where in the last step we have used that $\varphi(x + R_n 2^{-m}) = \varphi(x + R_m 2^{-m})$ due to the periodicity of φ . Hence, if we define

$$Z_n := \sum_{m=0}^{n-1} \beta_m Y_{n-m} = \sum_{m=1}^n \beta_{n-m} Y_m,$$

then

$$V_n = 2^n \mathbb{E}[\Phi(|2^{-n} Z_n|)]. \tag{4.4}$$

Thus, the Φ -variation of f is determined by the limit of the right-hand expectations as $n \uparrow \infty$.

Proof of Proposition 3.1 Taking $\Phi(x) = x^p$, we get from (4.4) that $V_n^{(p)} = 2^{n(1-p)} \mathbb{E}[|Z_n|^p]$. By Khintchine’s inequality, there exist constants A_p and B_p only depending on p such that $0 < A_p \leq B_p < \infty$, and

$$A_p 2^{n(1-p+pq_n)} = A_p 2^{n(1-p)} (s_n^2)^{p/2} \leq V_n^{(p)} \leq B_p 2^{n(1-p)} (s_n^2)^{p/2} = B_p 2^{n(1-p+pq_n)},$$

where $q_n := \frac{1}{n} \log_2 s_n$. From here, the assertion is straightforward. □

Now, we prove Theorem 3.3. For better accessibility, we have divided the proof into several parts.

Proof of Theorem 3.3 (a) Assuming that $\lim_n s_n^2 < \infty$, we define $\beta_m = \alpha_m 2^m$ and

$$\tilde{Z}_n := \sum_{m=0}^{n-1} \beta_m Y_m.$$

Then, the exchangeability of the sequence $\{Y_m\}_{m \in \mathbb{N}_0}$ implies that Z_n and \tilde{Z}_n have the same law. Clearly, $\tilde{Z}_n \rightarrow \tilde{Z} := \sum_{m=0}^{\infty} \beta_m Y_m$ in L^1 . Therefore, (4.4) yields for the choice $\Phi(x) := x$,

$$\sum_{k=0}^{2^n-1} |f((k+1)2^{-n}) - f(k2^{-n})| = 2^n \mathbb{E}[|2^{-n} Z_n|] \longrightarrow \mathbb{E}[|\tilde{Z}|].$$

Since f is continuous, this limit must coincide with the total variation of f (see, e.g., Theorem 2 in §5 of Chapter VIII in [22]). Conversely, parts (b) and (c) of Theorem 3.3 will imply that f can only be of finite total variation if $\lim_n s_n^2 < \infty$.

Proof of Theorem 3.3 (b) for $t = 1$ Here, we identify the Φ_0 -variation $\langle f \rangle_1^{\Phi_0}$ at time 1. The linearity of the Φ_0 -variation $t \mapsto \langle f \rangle_t^{\Phi_0}$ will be proved subsequently. Condition (3.4) for $q = 0$ together with our assumption $s_n^2 \rightarrow \infty$ implies via Proposition 1.5.1 in [3] that g is regularly varying with index $\varrho \geq 0$. Moreover, Remark 3.9, Lemma 2.2,

and Proposition 3.10 yield that the law and the first moment of Z_n/s_n converge, respectively, to $N(0, 1)$ and to its first moment.

Let us write $\psi(x) := \sqrt{g(x)}$ so that $2^n \Phi_0(2^{-n}|Z_n|) = |Z_n|/\psi(n - \log_2 |Z_n|)$. It follows that

$$2^n \Phi_0(2^{-n}|Z_n|) \mathbb{1}_{\{|Z_n|/\psi(n) \geq \varepsilon\}} = \frac{\psi(n)}{\psi(n - \log_2 |Z_n|)} \cdot \frac{|Z_n|}{\psi(n)} \mathbb{1}_{\{|Z_n|/\psi(n) \geq \varepsilon\}}. \tag{4.5}$$

To deal with the first factor on the right, we estimate $n - \log_2 |Z_n|$. First, on $\{|Z_n|/\psi(n) \geq \varepsilon\}$, we have $\log_2 |Z_n| \geq \log_2 \varepsilon + \ell(n)$, where $\ell(x) := \log_2 \psi(x)$ is slowly varying by Proposition 1.5.7 in [3] and hence satisfies $\ell(n)/n \rightarrow 0$ according to Proposition 1.3.6 (v) in [3]. Therefore, there exists $n_3 \in \mathbb{N}$ such that

$$n - \log_2 |Z_n| \leq n - \log_2 \varepsilon - \ell(n) \leq (1 + \varepsilon)n \quad \text{on } \{|Z_n|/\psi(n) \geq \varepsilon\} \quad \text{for all } n \geq n_3. \tag{4.6}$$

Second, to get a lower bound, we use that $|Z_n| \leq \sum_{m=0}^{n-1} |\beta_m|$. Jensen's inequality gives furthermore that

$$|Z_n| \leq \sum_{m=0}^{n-1} |\beta_m| \leq n \cdot \sqrt{\frac{1}{n} \sum_{m=0}^{n-1} \beta_m^2} = \sqrt{n} s_n. \tag{4.7}$$

By (3.4), there is $n_4 \in \mathbb{N}$ such that $s_n \leq (1 + \varepsilon)\psi(n)$ for all $n \geq n_4$. Thus, there is $n_5 \geq n_4$ such that

$$n - \log_2 |Z_n| \geq n - \log_2(1 + \varepsilon) - \frac{1}{2} \log_2 n - \ell(n) \geq (1 - \varepsilon)n \quad \text{for all } n \geq n_5. \tag{4.8}$$

Combining (4.6) and (4.8) now yields

$$\frac{\psi(n)}{\psi(n - \log_2 |Z_n|)} \geq \inf_{1 - \varepsilon \leq \lambda \leq 1 + \varepsilon} \frac{\psi(n)}{\psi(\lambda n)} \quad \text{for } n \geq n_5 \quad \text{on } \{|Z_n|/\psi(n) \geq \varepsilon\}. \tag{4.9}$$

According to the uniform convergence theorem for regularly varying functions (Theorem 1.5.2 in [3]) and the fact that ψ is regularly varying with index $\varrho/2 \geq 0$ by Proposition 1.5.7 (i) in [3], we have

$$\inf_{1 - \varepsilon \leq \lambda \leq 1 + \varepsilon} \frac{\psi(n)}{\psi(\lambda n)} \longrightarrow (1 - \varepsilon)^{\varrho/2}. \tag{4.10}$$

To deal with the second factor on the right-hand side of (4.5), we choose $n_6 \geq n_5$ such that $\psi(n) \leq (1 + \varepsilon)s_n$ for all $n \geq n_6$. Then, for $n \geq n_6$,

$$\frac{|Z_n|}{\psi(n)} \mathbb{1}_{\{|Z_n|/\psi(n) \geq \varepsilon\}} \geq \frac{|Z_n|}{(1 + \varepsilon)s_n} \mathbb{1}_{\{|Z_n|/s_n \geq \varepsilon(1 + \varepsilon)\}}. \tag{4.11}$$

Altogether, we get

$$\liminf_{n \uparrow \infty} 2^n \mathbb{E} [\Phi_0 (2^{-n} |Z_n|)] \geq \frac{(1 - \varepsilon)^{q/2}}{(1 + \varepsilon)\sqrt{2\pi}} \int_{\{|z| \geq \varepsilon(1+\varepsilon)\}} |z| e^{-z^2/2} dz.$$

With (4.4), we thus get $\liminf_n V_n \geq \sqrt{2/\pi}$ by sending $\varepsilon \downarrow 0$.

To get an upper bound, we recall that $x \mapsto \Phi_0(1/x)$ is regularly varying (at infinity) with index $-1 < 0$. Theorem 1.8.2 in [3] hence yields a function h that is regularly varying with index -1 , strictly decreasing, and satisfies $\Phi_0(1/x) \leq h(x)$ as well as $h(x)/\Phi_0(1/x) \rightarrow 1$ as $x \uparrow \infty$. Thus, the function $\Psi(x) := h(1/x)$ is strictly increasing, regularly varying at zero with index 1, and satisfies $\Phi_0 \leq \Psi$ as well as $\Psi(x)/\Phi_0(x) \rightarrow 1$ as $x \downarrow 0$. Let $\varepsilon > 0$ be given and choose $x_0 > 0$ such that $\Psi(x) \leq \Phi_0(x)/(1 - \varepsilon)$ for all $x \leq x_0$. By (4.7), there is $n_7 \in \mathbb{N}$ such that $2^{-n} |Z_n| \leq x_0$ for all $n \geq n_7$. For such n , we get

$$\begin{aligned} 2^n \mathbb{E} [\Phi_0 (2^{-n} |Z_n|)] &\leq 2^n \mathbb{E} [\Psi (2^{-n} |Z_n|)] \leq 2^n \mathbb{E} [\Psi (2^{-n} (\varepsilon + |Z_n|))] \\ &\leq \frac{2^n}{1 - \varepsilon} \mathbb{E} [\Phi_0 (2^{-n} (\varepsilon + |Z_n|))] \\ &= \frac{s_n}{(1 - \varepsilon)\psi(n)} \mathbb{E} \left[\frac{\varepsilon + |Z_n|}{s_n} \cdot \frac{\psi(n)}{\psi(n - \log_2(\varepsilon + |Z_n|))} \right]. \end{aligned}$$

Clearly, the factor $s_n/\psi(n)$ converges to 1. Moreover, by (4.8) there is $n_8 \geq n_5 \vee n_7$ such that

$$(1 - \varepsilon)n \leq n - \log_2(\varepsilon + |Z_n|) \leq n - \log_2 \varepsilon \leq (1 + \varepsilon)n \quad \text{for all } n \geq n_8.$$

As in (4.9) and (4.10), we thus get

$$\frac{\psi(n)}{\psi(n - \log_2(\varepsilon + |Z_n|))} \leq \sup_{1 - \varepsilon \leq \lambda \leq 1 + \varepsilon} \frac{\psi(n)}{\psi(\lambda n)} \rightarrow (1 + \varepsilon)^{q/2}.$$

We hence obtain the upper bound

$$\limsup_{n \uparrow \infty} V_n \leq \frac{(1 + \varepsilon)^{q/2}}{(1 - \varepsilon)\sqrt{2\pi}} \int |z| e^{-z^2/2} dz, \tag{4.12}$$

where the right-hand side reduces to $\sqrt{2/\pi}$ as $\varepsilon \downarrow 0$. □

The proof of Theorem 3.3 (b) for $0 < t < 1$ requires that we truncate the summation for V_n in (4.4) at those indices k for which $k2^{-n} \leq t$. This is equivalent to restricting the expectation in (4.4) to the set $\{2^{-n} R_n \leq t\}$. The following proof adapts the arguments of the proof of Theorem 3.3 (b) for $t = 1$ to this restricted case.

Proof of Theorem 3.3 (b) for $0 < t < 1$. Let $t \in (0, 1)$ be given. The Φ_0 -variation of f over the interval $[0, t]$ is equal to $\lim_n V_{n,t}$, where

$$\begin{aligned} V_{n,t} &:= \sum_{k=0}^{2^n-1} \Phi_0\left(|f((k+1)2^{-n}) - f(k2^{-n})|\right) \mathbb{1}_{[0,t]}(k2^{-n}) \\ &= 2^n \mathbb{E}\left[\Phi_0\left(2^{-n}|Z_n|\right) \mathbb{1}_{\{2^{-n}R_n \leq t\}}\right]. \end{aligned}$$

This can be proved similarly as in the derivation of (4.4), see e.g., [13]. We fix $\delta \in (0, t \wedge (1-t))$ and $m \in \mathbb{N}$ that $2^{-m} \leq \delta$. Recall that $R_m := \sum_{k=1}^m U_k 2^{k-1}$ and let us denote $R_{m,n} := R_n - R_{n-m}$ for short. Then,

$$\{2^{-n}R_{m,n} \leq t - \delta\} \subseteq \{2^{-n}R_n \leq t\} \subseteq \{2^{-n}R_{m,n} \leq t\}. \tag{4.13}$$

First, we derive a lower bound for $V_{n,t}$. To this end, we define for $0 \leq m < n$,

$$Z_{m,n} := \sum_{k=1}^{n-m} \beta_{n-k} Y_k = Z_n - \sum_{k=n-m+1}^n \beta_{n-k} Y_k.$$

Then, since $s_n \rightarrow \infty$,

$$\frac{|Z_n - Z_{m,n}|}{s_n} \leq \frac{1}{s_n} \sum_{k=0}^{m-1} |\beta_k| =: a_n \rightarrow 0,$$

and so the laws of $Z_{m,n}/s_n$ converge to $N(0, 1)$ in the L^1 -Wasserstein distance by Slutsky’s theorem. By combining (4.5), (4.9), (4.10), and (4.11), there is $n_9 \geq n_6 \vee m$ such that for all $n \geq n_9$, with $c_\varepsilon := (1 - \varepsilon)^{1+Q/2}/(1 + \varepsilon)$,

$$\begin{aligned} 2^n \Phi_0(2^{-n}|Z_n|) \mathbb{1}_{\{|Z_n|/\psi(n) \geq \varepsilon, 2^{-n}R_n \leq t\}} &\geq \frac{c_\varepsilon |Z_n|}{s_n} \mathbb{1}_{\{|Z_n|/\psi(n) \geq \varepsilon, 2^{-n}R_n \leq t\}} \\ &\geq \frac{c_\varepsilon |Z_{m,n}|}{s_n} \mathbb{1}_{\{|Z_n|/\psi(n) \geq \varepsilon, 2^{-n}R_n \leq t\}} \\ &\quad - \frac{c_\varepsilon |Z_n - Z_{m,n}|}{s_n} \\ &\geq \frac{c_\varepsilon |Z_{m,n}|}{s_n} \mathbb{1}_{\{|Z_n|/\psi(n) \geq \varepsilon, 2^{-n}R_n \leq t\}} \\ &\quad - \frac{c_\varepsilon \sum_{k=0}^{m-1} |\beta_k|}{s_n}, \end{aligned}$$

where the last two inequalities hold due to the triangle inequality. Again, the right-most term decays to zero as $n \uparrow \infty$. Hence, there is $n_{10} \geq n_9$ such that for $n \geq n_{10}$

and $|Z_{m,n}|/s_n \geq 2\varepsilon(1 + \varepsilon)$,

$$\frac{|Z_n|}{s_n} \geq \frac{|Z_{m,n}| - |Z_n - Z_{m,n}|}{s_n} \geq \frac{|Z_{m,n}| - \sum_{k=0}^{m-1} |\beta_k|}{s_n} \geq \varepsilon(1 + \varepsilon).$$

Hence, for those n and m , we have $\{|Z_{m,n}|/s_n \geq 2\varepsilon(1 + \varepsilon)\} \subseteq \{|Z_n|/s_n \geq \varepsilon(1 + \varepsilon)\}$ and in turn

$$\begin{aligned} V_{n,t} + c_\varepsilon a_n &\geq \mathbb{E} \left[\frac{c_\varepsilon |Z_{m,n}|}{s_n} \mathbb{1}_{\{|Z_n|/\psi(n) \geq \varepsilon, 2^{-n} R_n \leq t\}} \right] \\ &\geq \mathbb{E} \left[\frac{c_\varepsilon |Z_{m,n}|}{s_n} \mathbb{1}_{\{|Z_{m,n}|/s_n \geq 2\varepsilon(1+\varepsilon), 2^{-n} R_{m,n} \leq t-\delta\}} \right] \\ &\geq \mathbb{E} \left[\frac{c_\varepsilon |Z_{m,n}|}{s_n} \mathbb{1}_{\{|Z_{m,n}|/s_n \geq 2\varepsilon(1+\varepsilon)\}} \right] \cdot \mathbb{P}[2^{-n} R_n \leq t - \delta], \end{aligned}$$

where the last step follows from (4.13) and the independence of $Z_{m,n}$ and $R_{m,n}$. Clearly, $\mathbb{P}[2^{-n} R_n \leq t - \delta] \rightarrow t - \delta$ and

$$\mathbb{E} \left[\frac{c_\varepsilon |Z_{m,n}|}{s_n} \mathbb{1}_{\{|Z_{m,n}|/s_n \geq 2\varepsilon(1+\varepsilon)\}} \right] \rightarrow \frac{c_\varepsilon}{\sqrt{2\pi}} \int_{\{|z| \geq 2\varepsilon(1+\varepsilon)\}} |z| e^{-z^2/2} dz,$$

since $s_{n-m}/s_n \rightarrow 1$ by Proposition 3.10. Hence

$$\liminf_{n \uparrow \infty} V_{n,t} \geq \frac{c_\varepsilon(t - \delta)}{\sqrt{2\pi}} \int_{\{|z| \geq 2\varepsilon(1+\varepsilon)\}} |z| e^{-z^2/2} dz. \tag{4.14}$$

To obtain a corresponding upper bound, we can argue exactly as in the derivation of (4.12) to get

$$\begin{aligned} \limsup_{n \uparrow \infty} V_{n,t} &\leq (1 + \varepsilon)^{\ell/2} \limsup_{n \uparrow \infty} \mathbb{E} \left[\frac{\varepsilon + |Z_n|}{s_n} \mathbb{1}_{\{2^{-n} R_{m,n} \leq t\}} \right] \\ &= (1 + \varepsilon)^{\ell/2} \limsup_{n \uparrow \infty} \mathbb{E} \left[\frac{|Z_n|}{s_n} \mathbb{1}_{\{2^{-n} R_{m,n} \leq t\}} \right]. \end{aligned}$$

Using again the independence of $Z_{m,n}$ and $R_{m,n}$, we find that

$$\mathbb{E}[\mathbb{1}_{\{2^{-n} R_{m,n} \leq t\}}] \leq \mathbb{E}[|Z_{m,n}|] \cdot \mathbb{P}[2^{-n} R_{m,n} \leq t] + \sum_{k=0}^{m-1} |\beta_k|.$$

By (4.13), we have $\mathbb{P}[2^{-n} R_{m,n} \leq t] \leq \mathbb{P}[2^{-n} R_n \leq t + \delta] \rightarrow t + \delta$. Recall that $\frac{1}{s_n} \sum_{k=0}^{m-1} |\beta_k| \rightarrow 0$ as $n \uparrow \infty$. Next, the function $\ell(x) := g(\log_2 x)$ is slowly varying by Remark 3.9, and so

$$\frac{g(n)}{g(n - m)} = \frac{\ell(2^n)}{\ell(2^{-m} 2^n)} \rightarrow 1 \text{ as } n \uparrow \infty. \tag{4.15}$$

Thus, our condition (3.4) for $q = 0$ implies that

$$\mathbb{E} \left[\frac{|Z_{m,n}|}{s_n} \right] = \frac{s_{n-m}}{s_n} \mathbb{E} \left[\frac{|Z_{m,n}|}{s_{n-m}} \right] \rightarrow \sqrt{\frac{2}{\pi}} \text{ as } n \uparrow \infty.$$

Altogether, we conclude that $\limsup_n V_{n,t} \leq (1 + \varepsilon)^{e/2} (t + \delta) \sqrt{2/\pi}$. Combining this inequality with (4.14) and sending ε and δ to zero, we conclude the proof of the linearity of the Φ_0 -variation. \square

The proof of part (c) of Theorem 3.3 will be prepared with the following two lemmas.

Lemma 4.1 *Under the conditions of Theorem 3.3 (c), there is a constant K such that for all $\omega \in \Omega$ and all sufficiently large $n \in \mathbb{N}$,*

$$2^{-qn} |Z_n(\omega)| \leq K \sqrt{g(n)} \text{ and } |Z_n(\omega)| \leq K s_n.$$

Proof The function $\ell(x) := g(\log_2 x)$ is slowly varying by Remark 3.9, and so ℓ and the sequence $\{\beta_m\}_{m \in \mathbb{N}_0}$ satisfy the conditions of Theorem 2.1. Recall from (2.9) in the proof of that theorem that there exists $\delta_1 > 0$ such that for every $\delta \in (0, \delta_1)$ there is $n_1 \in \mathbb{N}$ such that for all $m \geq n_1$,

$$\begin{aligned} & 2^{qm} \sqrt{(1 - \delta)\ell(2^m) - (1 + \delta)\ell(2^{m-1})} 2^{-2q} \\ & < |\beta_m| < 2^{qm} \sqrt{(1 + \delta)\ell(2^m) - (1 - \delta)\ell(2^{m-1})} 2^{-2q}, \end{aligned}$$

and the arguments of the two square roots are positive. Now let $\delta \in (0, \delta_1)$ be given and take $n_2 \geq n_1$ such that $1 - \delta < g(n)/g(n - 1) < 1 + \delta$ for any $n \geq n_2$; this is possible by (4.15). Since g is regularly varying, we have $2^{2qn} g(n) \rightarrow \infty$ as $n \uparrow \infty$ (see, e.g., Proposition B.1.9 in [5]). Therefore, we may apply the Stolz–Cesàro theorem in its general form so as to obtain

$$\begin{aligned} \limsup_{n \uparrow \infty} \frac{\sum_{m=0}^n |\beta_m|}{2^{qn} \sqrt{g(n)}} & \leq \limsup_{n \uparrow \infty} \frac{|\beta_n|}{2^{qn} \sqrt{g(n)} - 2^{q(n-1)} \sqrt{g(n-1)}} \\ & \leq \limsup_{n \uparrow \infty} \frac{\sqrt{(1 + \delta)g(n) - (1 - \delta)g(n-1)} 2^{-2q}}{\sqrt{g(n)} - \sqrt{2^{-2q}g(n-1)}} \\ & \leq \frac{\sqrt{(1 + \delta)^2 2^{2q} - (1 - \delta)}}{\sqrt{2^{2q}(1 - \delta) - 1}}. \end{aligned}$$

Since $|Z_n| \leq \sum_{m=0}^{n-1} |\beta_m|$, the claim follows. \square

Lemma 4.2 *Suppose that $\{z_n\}_{n \in \mathbb{N}_0}$ is a sequence of nonnegative numbers converging to $z > 0$. Then $2^n \Phi_q(2^{-n} s_n z_n) \rightarrow z^{1/(1-q)}$ as $n \uparrow \infty$.*

Proof We may assume without loss of generality that $z_n > 0$ for all n . Then, we may write

$$\begin{aligned} 2^n \Phi_q(2^{-n} s_n z_n) &= 2^n (2^{-n} s_n z_n)^{1/(1-q)} g\left(\frac{n - \log_2 s_n - \log_2 z_n}{1 - q}\right)^{-1/(2(1-q))} \\ &= \left(\frac{s_n}{2^{qn} \sqrt{g(n)}}\right)^{1/(1-q)} z_n^{1/(1-q)} \left(\frac{g(n)}{g\left(\frac{n - \log_2 s_n - \log_2 z_n}{1 - q}\right)}\right)^{1/(2(1-q))}. \end{aligned}$$

By (3.4), the first factor on the right converges to 1. The second factor converges to $z^{1/(1-q)}$ by assumption. To deal with the third factor, (3.4) implies that there is $n_0 \in \mathbb{N}$ such that

$$qn - \log_2 \sqrt{g(n)} - 1 \leq \log_2 s_n \leq qn - \log_2 \sqrt{g(n)} + 1 \quad \text{for all } n \geq n_0.$$

Theorem 1.4.1 and Proposition 1.3.6 (i) in [3] give moreover that there exists $\kappa^+, \kappa^- \in \mathbb{R}$ such that $\kappa^- \log_2 n \leq \log_2 \sqrt{g(n)} \leq \kappa^+ \log_2 n$. Thus, if $n \geq n_0$ is sufficiently large such that $z/2 \leq z_n \leq 2z$, then

$$\begin{aligned} n + \frac{-1 - \log_2(2z) - \kappa^+ \log_2 n}{1 - q} &\leq \frac{n - \log_2 s_n - \log_2 z_n}{1 - q} \\ &\leq n + \frac{1 - \log_2(z/2) - \kappa^- \log_2 n}{1 - q}. \end{aligned} \tag{4.16}$$

For sufficiently large n , the center term can thus be expressed as $n\lambda_n$, where $\lambda_n \in [1/2, 3/2]$ and $\lambda_n \rightarrow 1$. The uniform convergence theorem for regularly varying functions (e.g., Theorem 1.5.2 in [3]) hence implies that

$$\frac{g(n)}{g\left(\frac{n - \log_2 s_n - \log_2 z_n}{1 - q}\right)} \rightarrow 1$$

as $n \uparrow \infty$. This concludes the proof. □

Proof of Theorem 3.3 (c) We prove part (c) only for $t = 1$. The extension to $0 < t < 1$ is almost verbatim identical to the one in part (b) and hence left to the reader. We write

$$\widehat{Z}_n := \frac{1}{s_n} \sum_{m=1}^n |\beta_{n-m}| Y_m.$$

Then the law of $2^n \Phi_q(2^{-n} |Z_n|)$ is the same as that of $2^n \Phi_q(2^{-n} s_n |\widehat{Z}_n|)$. By Theorem 2.1, we have $|\widehat{Z}_n| \rightarrow |Z|$ in L^∞ . Moreover, by either Corollary 6.6 or Remark 6.7 in [6], the law of Z has no atoms. In particular, we have $\mathbb{P}[Z = 0] = 0$. Lemma 4.2

hence yields that \mathbb{P} -a.s. $2^n \Phi_q(2^{-n} s_n |\widehat{Z}_n|) \rightarrow |Z|^{1/(1-q)}$. Fatou's lemma thus gives immediately that

$$\begin{aligned} \liminf_{n \uparrow \infty} V_n &= \liminf_{n \uparrow \infty} 2^n \mathbb{E}[\Phi_q(2^{-n} |Z_n|)] = \liminf_{n \uparrow \infty} \mathbb{E}[2^n \Phi_q(2^{-n} s_n |\widehat{Z}_n|)] \\ &\geq \mathbb{E}[|Z|^{1/(1-q)}]. \end{aligned}$$

To get an upper bound, we argue as in the proof of Theorem 3.3 (b) for $t = 1$ that there exists an increasing function Ψ that is regularly varying at zero with index $1/(1 - q)$ and satisfies $\Phi_q(x) \leq \Psi(x)$ for all sufficiently small x as well as $\Psi(x)/\Phi_q(x) \rightarrow 1$ as $x \downarrow 0$. In particular, we have $2^n \Psi(2^{-n} s_n z_n) \rightarrow z^{1/(1-q)}$ if $\{z_n\}_{n \in \mathbb{N}_0}$ is a sequence of nonnegative numbers converging to $z > 0$. Let $\varepsilon > 0$ be given. We choose $x_0 > 0$ such that $\Psi(x) \leq \Phi_q(x)/(1 - \varepsilon)$ for all $x \leq x_0$. Since the sequence $\{\widehat{Z}_n\}$ is uniformly bounded, there is $n_0 \in \mathbb{N}$ such that $2^{-n} s_n (\varepsilon + |\widehat{Z}_n|) \leq x_0$ \mathbb{P} -a.s. for all $n \geq n_0$. For such n , we hence have

$$\begin{aligned} V_n &= 2^n \mathbb{E}[\Phi_q(2^{-n} s_n |\widehat{Z}_n|)] \leq 2^n \mathbb{E}[\Psi(2^{-n} s_n |\widehat{Z}_n|)] \leq 2^n \mathbb{E}[\Psi(2^{-n} s_n (\varepsilon + |\widehat{Z}_n|))] \\ &\leq \frac{2^n}{1 - \varepsilon} \mathbb{E}[\Phi_q(2^{-n} s_n (\varepsilon + |\widehat{Z}_n|))]. \end{aligned}$$

Moreover, we see from (4.16) that there exists $n_1 \geq n_0$ such that the random variables Λ_n defined through

$$\Lambda_n := \frac{n - \log_2 s_n - \log_2(\varepsilon + |\widehat{Z}_n|)}{n(1 - q)}$$

take values in $[1/2, 3/2]$ for $n \geq n_1$ and satisfy $\Lambda_n \rightarrow 1$ \mathbb{P} -a.s. Hence, with ϱ denoting the index of regular variation of g ,

$$0 \leq \frac{g(n)}{g\left(\frac{n - \log_2 s_n - \log_2(\varepsilon + |\widehat{Z}_n|)}{1 - q}\right)} = \frac{g(n)}{g(n \Lambda_n)} \leq \sup_{1/2 \leq \lambda \leq 3/2} \frac{g(n)}{g(n \lambda)} \rightarrow \left(\frac{3}{2}\right)^\varrho$$

according to the uniform convergence theorem for regularly varying functions (e.g., Theorem 1.5.2 in [3]). Hence, Lemma 4.2 and dominated convergence give that

$$\limsup_{n \uparrow \infty} V_n \leq \limsup_{n \uparrow \infty} \frac{1}{1 - \varepsilon} 2^n \mathbb{E}[\Phi_q(2^{-n} s_n (\varepsilon + |\widehat{Z}_n|))] = \frac{1}{1 - \varepsilon} \mathbb{E}[(\varepsilon + |Z|)^{1/(1-q)}].$$

Sending $\varepsilon \downarrow 0$ gives the desired upper bound. □

Proof of Corollary 3.5 (a) Taking logarithms in (3.4) and using the fact that $\frac{1}{n} \log_2 g(n) \rightarrow 0$, we get $\frac{1}{n} \log_2 s_n \rightarrow q$ as $n \uparrow \infty$. Hence, the result follows from Proposition 3.1.

- (b) If $0 < c < \infty$, then (3.4) holds also if g is replaced with $\tilde{g}(x) := c$. Hence, the assertion follows immediately from Theorem 3.3. If $c = 0$, then for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$g\left(\frac{-\log_2 \eta}{1-q}\right) < \varepsilon \quad \text{for all } \eta < \delta.$$

Now it suffices to take $n_0 \in \mathbb{N}$ such that $|f((k+1)2^{-n}) - f(k2^{-n})| < \delta$ for all k and $n \geq n_0$ to obtain that

$$\sum_{k=0}^{\lfloor t2^n \rfloor} |f((k+1)2^{-n}) - f(k2^{-n})|^p \leq \varepsilon^{p/2} \sum_{k=0}^{\lfloor t2^n \rfloor} \Phi_q(|f((k+1)2^{-n}) - f(k2^{-n})|).$$

Sending $\varepsilon \downarrow 0$ gives the result. The case $c = \infty$ is obtained analogously. \square

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