Sample Path Properties of the Fractional Wiener–Weierstrass Bridge

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Abstract

Fractional Wiener–Weierstrass bridges are a class of Gaussian processes that arise from replacing the trigonometric function in the construction of classical Weierstrass functions by a fractional Brownian bridge. We investigate the sample path properties of such processes, including local and uniform moduli of continuity, Φ-variation, Hausdorff dimension, and location of the maximum. Our analysis relies heavily on upper and lower bounds of fractional integrals, where we establish a novel improvement of the classical Hardy–Littlewood inequality for fractional integrals of a special class of step functions.

Keywords: Fractional Wiener–Weierstrass bridge, moduli of continuity, Φ-variation, Hausdorff dimension, Hardy–Littlewood inequality for fractional integrals

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1 Introduction

In recent years, interest in non-diffusive stochastic models—those with sample paths either rougher or smoother than standard Brownian motion—has grown significantly. This trend is driven by several factors. First, advances in Lyons' rough path theory [11], modeling with fractional Brownian motion [27], and pathwise Itô calculus for trajectories with p-th variation for $p > 2$ [7] have all spurred further research. Second, new applications of non-diffusive processes, such as rough volatility modeling [4], have underscored their practical relevance.

The phenomenon of roughness has also played a significant role in fractal geometry. Consider, for instance, the classical Weierstrass function, which, for $\alpha \in (0, 1)$ and $b \in \{2, 3, \dots\}$, is defined as

$$\phi_{\alpha,b}(t) = \sum_{n=0}^{\infty} \alpha^n \cos(2\pi b^n t), \qquad t \in [0, 1]. \tag{1}$$

Recent analyses have studied this function in a manner analogous to the sample paths of stochastic processes. Among the properties explored are the local and uniform moduli of continuity [8], the existence of a local time [19], the Hausdorff dimension of its graph [25], and its suitability as an integrator for pathwise Itô calculus [39].

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In this paper, we continue the analysis of a new class of stochastic processes introduced by the authors in [40], aiming to provide a synthesis between fractional Gaussian processes and fractal geometry. These processes are obtained by replacing the cosine function in (1) by the trajectories of a fractional Brownian bridge B_H with Hurst parameter H (the precise definition of B_H is given by (3) below). More precisely, the *fractional Wiener–Weierstrass bridge* with parameters $\alpha \in (0,1)$, $b \in \{2,3,\dots\}$, and $H \in (0,1)$ is defined as the stochastic process

$$Y(t) := \sum_{n=0}^{\infty} \alpha^n B_H(\{b^n t\}), \qquad 0 \le t \le 1,$$

where $\{x\}$ is the fractional part of $x \ge 0$. Although Y remains a Gaussian process, it displays a number of intriguing properties. For instance, it was shown in [40] that there is no combination of parameter values (α, b, H) for which Y is a semimartingale. It was shown moreover that the covariance function $c(s,t) := \mathbb{E}[Y(s)Y(t)]$ typically has a fractal structure and, for fixed $s \in (0,1)$, can have the same roughness as the sample paths of Y. Here, the roughness of a function $f : [0,1] \to \mathbb{R}$ is quantified through its roughness exponent along the b-adic partitions, which according to [17] is defined as a number $R \in [0,1]$ for which

$$\lim_{n\uparrow\infty} \sum_{k=0}^{b^n-1} \big|f((k+1)b^{-n}) - f(kb^{-n})\big|^p = \begin{cases} \infty & \text{if } p < 1/R; \\ 0 & \text{if } p > 1/R. \end{cases} \tag{2}$$

The main results of [40] identify the roughness exponent R of the sample paths of Y and compute their p-th variation (i.e., the limit on the left-hand side of (2)) for $p = 1/R$. These results show that two distinct regimes emerge, arising from the competition between the roughness exponents of the trajectories of the fractional Brownian bridge B_H and the classical Weierstrass function in (1), which are given by H and $K := 1 \wedge (-\log_b \alpha)$, respectively.

In this paper, we continue our analysis of the fractional Wiener–Weierstrass bridge, focusing on several sample path properties: (Wiener–Young) Φ-variation, the local and uniform moduli of continuity, the Hausdorff dimension of the graph, and the location of the maximum. Our results highlight the significance of the distinct regimes arising from the relationship between the roughness exponents H and K. Specifically, for $H < K$, we observe that the fine structure of the sample paths retains characteristics similar to those of B_H. In contrast, when $K < H$, the behavior of the trajectories of Y resembles that of a randomized version of the classical Weierstrass function. In this case, the limits of the local and uniform moduli of continuity are governed by non-degenerate random variables, providing an example for which the well-known zero-one law for the moduli of continuity of Gaussian processes fails (see Lemma 7.1.1 of [29] or Lemma 4.6 below). The critical case $H = K$ presents the most intriguing and challenging regime. Here, the fine structure of the fractional Wiener–Weierstrass bridge deviates from both the classical Weierstrass function and the trajectories of B_H.

There is extensive literature on the sample path properties of Gaussian processes; see, for example, the books by Adler [2] and Marcus and Rosen [29]. However, most proofs in this area rely on the stationarity of increments and explicit knowledge of the covariance function. In contrast, the fractional Wiener–Weierstrass process has highly non-stationary increments, a fractal covariance structure, and lacks self-similarity, posing significant challenges for our analysis. To handle the complex covariance structure of Y, we establish upper and lower bounds on $\mathbb{E}[(Y(t) - Y(s))^2]$ for sufficiently small $|t - s|$. These bounds depend on a novel extension of the classical Hardy–Littlewood inequality for fractional integrals, presented in Theorem 3.3. The non-stationarity of the increments requires us to refine traditional methods for deriving sample path properties of Gaussian processes. For example, in Theorem 4.1, we establish a general result on the Φ-variation of Gaussian processes, extending Theorem

4 from [24] to processes with non-stationary increments. Additionally, we employ classic techniques such as strong local non-determinism, the Sudakov minoration, and the concentration of measure.

The rest of this paper is organized as follows. In Section 2, we state our main results, beginning with Theorem 2.2 on the Φ-variation of the sample paths, followed by Theorem 2.3 and 2.4, which address the local and uniform moduli of continuity. In Theorem 2.6, we determine the Hausdorff dimension of the graph of Y as $(2-H)\vee(2-K)$, assuming that $K>2H-1$. Theorem 2.7 establishes that the location of the maximum of Y has an atomless distribution if and only if $H>K$. Section 2 concludes with an outlook and a list of open questions. Section 3 provides auxiliary results, some of which may be of independent interest—particularly the extended Hardy–Littlewood inequality given by Theorem 3.3, which plays a central role in our proofs. Finally, Section 4 contains the proofs of our main results.

2 Main results

Following [40], let $W_H=(W_H(t))_{t\geq 0}$ be the fractional Brownian motion with Hurst parameter $H\in(0,1)$ and starting point $W_H(0)=0$. We pick a deterministic function $\kappa:[0,1]\to[0,1]$ satisfying $\kappa(0)=0$ and $\kappa(1)=1$. The stochastic process

$$B_H(t):=W_H(t)-\kappa(t)W_H(1),\qquad t\in[0,1],\tag{3}$$

can then be regarded as a fractional Brownian bridge with Hurst parameter H. For instance, under the choice

$$\kappa(t):=\frac{1}{2}(1+t^{2H}-(1-t)^{2H}),\tag{4}$$

the law of B_H is equal to the law of W_H conditioned on the event $\{W_H(1)=0\}$; see [16]. However, the specific form of κ will not be needed in the sequel. We will only assume that B_H is of the form $B_H(t)=W_H(t)-\kappa(t)W_H(1)$ for some function $\kappa:[0,1]\to[0,1]$ that satisfies $\kappa(0)=0$ and $\kappa(1)=1$ and that is Hölder continuous with some exponent $\tau\in(H,1]$. For example, the function κ in (4) satisfies these requirements. Both κ and B_H will be fixed throughout this paper. We denote by $\{x\}$ the fractional part of $x\geq 0$.

Definition 2.1. ([40]) For $\alpha\in(0,1)$ and $b\in\{2,3,\dots\}$, the stochastic process

$$Y(t)=Y_{\alpha,b,H}(t):=\sum_{n=0}^{\infty}\alpha^n B_H(\{b^n t\}),\qquad 0\leq t\leq 1,\tag{5}$$

is called the *fractional Wiener–Weierstrass bridge* with parameters α, b, and H.

The fractional Wiener–Weierstrass bridge is a Gaussian process with highly non-stationary increments and, unlike fractional Brownian motion, is not self-similar. Therefore, many techniques for studying sample path properties are not available. In addition, it has continuous but nowhere-differentiable sample paths, a fractional covariance structure, and is not a semimartingale [40].

Throughout this paper, we define

$$K:=\min\Big\{1,\big(-\log_b\alpha\big)\Big\}.\tag{6}$$

The key message from [40] is the following.

The roughness of the fractional Wiener–Weierstrass bridge $Y = (Y(t))_{t\in[0,1]}$ is determined by a competition between the Hurst exponent H of the underlying fractional Brownian bridge and the roughness exponent K from the Weierstrass-type convolution.

In other words, the process Y often inherits sample path properties from the fractional Brownian motion W_H if $H < K$, and from the Weierstrass fractal construction if $H > K$. Therefore, the investigation of the sample path properties of Y often bifurcates, depending on the relation between K and H. For instance, the *roughness* of a function $f : [0, 1] \to \mathbb{R}$ can be quantified by the p-th variation along the sequence of b-adic partitions, defined as

$$\langle f\rangle_t^{(p)} := \lim_{n\uparrow\infty} \sum_{k=0}^{\lfloor tb^n\rfloor} \left|f((k+1)b^{-n}) - f(kb^{-n})\right|^p, \qquad t \in [0, 1], \tag{7}$$

provided the limit exists for all t, where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x (see [17]). Theorem 2.3 of [40] shows that the p-th variation of Y along the sequence of b-adic partitions is non-trivial for $p = \min\{1/K, 1/H\}$ if $K \neq H$. We refer to Section 2 of [40] for further discussions and properties of the fractional Wiener–Weierstrass bridge.

Our results on the sample path properties of the process Y will also depend on the interplay between the parameters H and K. The most interesting and challenging case is the critical phase $H = K$, where we fully characterize the sample path properties through delicate analysis of the covariance function. We start from the (Wiener–Young) Φ-variation, which for a real function f is defined as

$$v_\Phi(f) = \sup\left\{\sum_{i=1}^{n} \Phi(|f(t_i) - f(t_{i-1})|) : \ 0 = t_0 < t_1 < \cdots < t_n = 1, \ n \in \mathbb{N}\right\}.$$

Note that here the supremum is taken over *all* partitions of the interval $[0, 1]$ and not just over a specific refining sequence of partitions as for the p^{th} variation studied in [40]. Our goal is to find a critical function Φ such that $v_\Phi(Y)$ is non-trivial in the sense that $\mathbb{P}(0 < v_\Phi(Y) < \infty) = 1$. Here and in the sequel, L (resp. $\delta > 0$) will denote a large (resp. small) number depending only on α, b, H and which may vary at each occurrence. The function $\log\log x$ is always interpreted as $\log\log x \vee e$, where $a \vee b$ and $a \wedge b$ denote the respective maximum and minimum of two real numbers a and b.

Theorem 2.2. *Let Y be a fractional Wiener–Weierstrass bridge with parameters α, b, and H, and $K = \min\{1, (-\log_b \alpha)\}$.*

(i) If $H < K$,

$$\mathbb{P}\left(\frac{1}{L} < v_\Phi(Y) < \infty\right) = 1 \quad \textit{for} \quad \Phi(x) = \left(\frac{x}{\sqrt{2\log\log(1/x)}}\right)^{1/H}. \tag{8}$$

(ii) If $H = K$,

$$\mathbb{P}\left(\frac{1}{L} < v_\Phi(Y) < \infty\right) = 1 \quad \textit{for} \quad \Phi(x) = \left(\frac{x}{\sqrt{2\log(1/x)\log\log(1/x)/H}}\right)^{1/H}. \tag{9}$$

(iii) If $H > K$,

$$\mathbb{P}(0 < v_\Phi(Y) < \infty) = 1 \quad \textit{for} \quad \Phi(x) = x^{1/K}.$$

Moreover, in all three cases, if $\Theta : [0,\infty) \to [0,\infty)$ *is a function such that* $\Phi(x) = o(\Theta(x))$ *as* $x \to 0^+$, *then* $v_\Theta(Y) = \infty$ *almost surely. Conversely, if* $\Theta(x) = o(\Phi(x))$ *as* $x \to 0^+$, *then* $v_\Theta(Y) = 0$ *almost surely.*

Theorem 2.2 will follow from a more general result, Theorem 4.1 below, which extends Theorem 10.3.2 of [29] to the case of Gaussian processes with non-stationary increments. The concept of Φ-variation plays an important role in rough path calculus; see, e.g., [11, 12], and the references therein. The Φ-variation of various stochastic processes has been the subject of several earlier works: fractional Brownian motion [9], bi-fractional Brownian motion [34], sub-fractional Brownian motion [37], more general Gaussian processes with stationary increments [24], and certain non-Gaussian processes [3]. For fractional Brownian motion, a critical function Φ such that $v_\Phi(W_H)$ is non-trivial is given by $\Phi(x) = x^{1/H}(\log\log(1/x))^{-1/(2H)}$, which coincides with the function Φ in part (i) of Theorem 2.2.

Our next results, Theorems 2.3 and 2.4, characterize the local and uniform moduli of continuity for fractional Wiener–Weierstrass bridges. We will write $\mathbb{R}_+ = (0,\infty)$. Following Section 7.1 of [29], we will say that a function $\omega : [0,1) \to [0,\infty)$ with $\omega(0) = 0$ is an *exact uniform modulus of continuity* for a Gaussian process $(G(t))_{t\in[0,1]}$ if

$$\mathbb{P}\left(\lim_{h\to 0} \sup_{\substack{t,s\in[0,1]\\ |t-s|<h}} \frac{|G(t)-G(s)|}{\omega(|t-s|)} = C\right) = 1$$

for some constant $C \in \mathbb{R}_+$. We say that a function $\rho : [0,1) \to [0,\infty)$ with $\rho(0) = 0$ is an *exact local modulus of continuity* for $(G(t))_{t\in[0,1]}$ at $s \in [0,1]$ if

$$\mathbb{P}\left(\limsup_{t\in[0,1],\, t\to s} \frac{|G(t)-G(s)|}{\rho(|t-s|)} = C'\right) = 1$$

for some $C' \in \mathbb{R}_+$.

Theorem 2.3. *Let* Y *be a fractional Wiener–Weierstrass bridge with parameters* α, b, *and* H, *and* $K = \min\{1, (-\log_b \alpha)\}$.

(i) If $H < K$, *then* $\rho(u) = u^H\sqrt{\log\log(1/u)}$ *is an exact local modulus of continuity for* Y *at every* $s \in [0,1]$.

(ii) If $H = K$, *then* $\rho(u) = u^H\sqrt{\log(1/u)\log\log(1/u)}$ *is an exact local modulus of continuity for* Y *at every* $s \in [0,1]$.

(iii) If $H > K$, *then there exist non-negative random variables* Z_s, $s \in [0,1]$, *such that*

$$\mathbb{P}\left(\limsup_{t\in[0,1],\, t\to s} \frac{|Y(t)-Y(s)|}{|t-s|^K} = Z_s\right) = 1. \tag{10}$$

Moreover, if κ *is strictly increasing,*[1] *the random variable* Z_s *is non-constant, strictly positive, and unbounded for almost every* $s \in [0,1]$. *In particular,* Y *does not have an exact local modulus of continuity at almost every* $s \in [0,1]$.

Theorem 2.4. *Let* Y *be a fractional Wiener–Weierstrass bridge with parameters* α, b, *and* H, *and* $K = \min\{1, (-\log_b \alpha)\}$.

[1] This is a technical assumption, which we expect can be removed.

(i) If $H < K$, then $\omega(u) = u^H \sqrt{\log(1/u)}$ is an exact uniform modulus of continuity for Y.

(ii) If $H = K$, then $\omega(u) = u^H \log(1/u)$ is an exact uniform modulus of continuity for Y.

(iii) If $H > K$, then there exists a non-constant and unbounded random variable Z such that

$$\mathbb{P}\left(\lim_{h\to 0} \sup_{\substack{t,s\in[0,1] \\ |t-s|<h}} \frac{|Y(t) - Y(s)|}{|t-s|^K} = Z \right) = 1. \tag{11}$$

In particular, Y does not have an exact uniform modulus of continuity.

It is instructive to compare the moduli of continuity of Y with those of classical functions or processes. First, if $\alpha b > 1$, the classical Weierstrass function (1) admits local moduli $\rho(u) = u^{-\log_b(\alpha)}$ at all points (Theorem 1 of [20]), and hence the same uniform modulus. Second, the fractional Brownian motion W_H has an exact uniform modulus $\omega(u) = u^H \sqrt{\log(1/u)}$ and an exact local modulus $\rho(u) = u^H \sqrt{\log\log(1/u)}$, as special cases of Theorems 7.2.14 and 7.6.4 of [29]. In other words, in terms of moduli of continuity, the Wiener–Weierstrass process Y mimics the classical Weierstrass-type functions if $H > K$ and the fractional Brownian motion W_H if $H < K$. Surprisingly, in the critical case $H = K$, our modulus of continuity differs from that of the critical Weierstrass function (1) with $\alpha b = 1$. More precisely, [15] established that $\rho(u) = u^H \sqrt{\log(1/u) \log\log\log(1/u)}$ is an exact local modulus of continuity for $\phi_{1/b,b}$ at almost all points. More general results were later established in [8] for (critical) Weierstrass-type functions.

Remark 2.5. The fact that Z_s in (10) (or Z in (11)) is random does not contradict the well-known zero-one law on the modulus of continuity of Gaussian processes (see Lemma 4.6 below). In fact, it is an interesting example where the zero-one law fails. This is because for $H > K$ the modulus of continuity is *not* of a larger order than the L^2-distance $\|Y(t) - Y(s)\|_2$ as $|t-s| \to 0$.

Denote by $\dim(f)$ the Hausdorff dimension of the graph of a function f. For the Weierstrass function in (1) (or Weierstrass-type functions in general), it was a long-standing conjecture that $\dim(\phi_{\alpha,b}) = \max\{1, 2-K\}$, until recently resolved in [38, 41]. On the other hand, for the fractional Brownian motion we have $\dim(W_H) = 2 - H$ a.s. (Theorem 1 of [1]). The question extends naturally to the fractional Wiener–Weierstrass bridge Y. According to the previous heuristic, it is expected that $\dim(Y) = \max\{2-H, 2-K\}$ holds a.s. We give a partial answer in the next result.

Theorem 2.6. *Let Y be a fractional Wiener–Weierstrass bridge with parameters α, b, and H, and $K = \min\{1, (-\log_b \alpha)\}$. Suppose that $K > 2H - 1$. Then $\dim(Y) = \max\{2-H, 2-K\}$ almost surely.*

A crucial ingredient in analyzing the moduli of continuity and the Hausdorff dimension is estimating the covariance of the fractional Wiener–Weierstrass bridge Y. Specifically, we need to prove upper and lower bounds for $\|Y(t) - Y(s)\|_2$ for $t, s \in [0,1]$. This non-trivial task will require two technical tools: fractional integral representations and the strong local-nondeterminism property. We will provide the necessary background on fractional integrals in Section 3.1.

The strong local non-determinism is a crucial property for establishing sample path properties of Gaussian processes, such as uniform modulus of continuity [30], small ball probability, and Chung's law of the iterated logarithm [43], among many others. See [45, 46] for surveys. There are multiple different definitions of local non-determinism for Gaussian processes, among which [5] and [36] contain the earliest versions. One of the most widely used definitions is as follows: we say a Gaussian random field $(X(t))_{t\in I}$ indexed by $I \subseteq \mathbb{R}^N$ is strongly locally ρ-non-deterministic if, for any $n \in \mathbb{N}$ and any points $u, t_1, \ldots, t_n \in I$,

$$\operatorname{Var}\big(X(u) \mid X(t_1), \ldots, X(t_n)\big) \geq \frac{1}{L} \min_{1\leq k\leq n} \rho(u, t_k)^2, \tag{12}$$

where ρ is some prefixed metric on $\mathbb{R}^N$ and $L > 0$ depends only on the law of the process $(X(t))_{t\in I}$. For instance, one may take $\rho(s,t) = |t-s|^H$ for the fractional Brownian motion (Lemma 7.1 of [36]). This property also holds for well-known Gaussian processes such as the fractional Brownian sheet [44] and bi-fractional Brownian motion [43], with certain choices of ρ.

Finally, we study the location of the maximum of a fractional Wiener–Weierstrass bridge. Let $\tau_{\alpha,b,H}$ be the (random) location of the maximum of $Y_{\alpha,b,H}$, taking the leftmost point when the location is not unique, that is,

$$\tau_{\alpha,b,H} := \min\Big\{t \in [0,1] : Y_{\alpha,b,H}(t) = \sup_{0\leq s\leq 1} Y_{\alpha,b,H}(s)\Big\}.$$

Theorem 2.7. *Let $K = \min\{1, (-\log_b \alpha)\}$. The distribution of $\tau_{\alpha,b,H}$ is atomless if and only if $H \leq K$. Moreover, $\mathbb{P}(\tau_{\alpha,b,H} = 0) > 0$ if $H > K$.*

The study of the location of the maximum of (deterministic) fractal functions has extensive literature. To mention a few, there are [21] for the classical Takagi function, [13, 14, 32] for Takagi–Landsberg functions, and [18] that characterizes the set of global maximizers and minimizers for functions in the Takagi class.

Notations. We use $\|\cdot\|_2$ to denote the $L^2(\Omega)$-norm of a generic random variable and $\|\cdot\|_{L^p}$ to denote the L^p-norm of a measurable function in $L^p(\mathbb{R})$, where $p \geq 1$. Denote by $\#A$ the cardinality of a finite set A, and $|I|$ the Lebesgue measure of a set $I \subseteq \mathbb{R}$. The set of non-negative integers is denoted by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Outlook and some open questions. We propose a few open questions and interesting directions for future research.

1. Several proofs (such as that of Theorem 2.4(ii)) could be simplified, and further properties of the fractional Wiener–Weierstrass bridges could perhaps be unveiled, if one can establish the strong local non-determinism (12) for the Wiener–Weierstrass processes. However, this task appears non-trivial due to the intricate dependence structure created by the fractal construction. We conjecture that if $H < K$, then Y is strongly locally ρ-non-deterministic with $\rho(s,t) = |s-t|^H$.

2. As mentioned above, our main results rely heavily on controlling the L^2-distance $\|Y(t) - Y(s)\|_2$. As we will see in Section 3.3, obtaining precise bounds of $\|Y(t) - Y(s)\|_2$ in the case $H > K$ remains a challenging task. On one hand, it is not hard to show that $\|Y(t) - Y(s)\|_2 \leq L|t-s|^K$ uniformly in s, t. On the other hand, a lower bound of the form

$$\|Y(t) - Y(s)\|_2 \geq \frac{1}{L}|t-s|^K \tag{13}$$

cannot hold uniformly for $s, t \in [0,1]$, because otherwise it would contradict Lemma 7.1.10 of [29] along with Theorem 2.3(iii). This motivates the following question: for which pairs $(s,t) \in [0,1]^2$ can we have a uniform lower bound (13)? Lemma 3.9 may serve as a first step by asserting that if

$K \in (2H-1, H)$, there exist sets $\{T_N\}_{N\geq 1}$ with Hausdorff dimensions tending to one, such that (13) holds uniformly for $t, s \in T_N$, where L may depend on N. Extensions of such a result may lead to a better understanding of the local modulus of continuity and the Hausdorff dimension; see the point below.

3. We conjecture that for all α, b, H, it holds that $\dim(Y) = \max\{2-H, 2-K\}$ almost surely. Moreover, we conjecture that the random variable Z_s arising in Theorem 2.3(iii) is non-constant and strictly positive for all $s \in [0,1]$. For this problem, the case $H \in (0, 1/2]$ should follow from the Hardy–Littlewood inequality (Lemma 3.1 below), while the more challenging case is $H \in (1/2, 1)$. Both problems require further investigation of the quantity $\|Y(t) - Y(s)\|_2$, particularly the lower bound.

3 Some preliminary estimates

An essential ingredient in proving the main results is estimating the covariance of the fractional Wiener–Weierstrass bridge Y, or in other words, obtaining upper and lower bounds for $\|Y(t)-Y(s)\|_2$. This will be the goal of the current section. We start with a minimal background on fractional integrals and their connections to moments of fractional Wiener integrals.

3.1 Background on fractional integration

Let us recall from [31] that the Riemann–Liouville fractional integrals are defined as follows. For $\beta > 0$,

$$I_+^{\beta}(f)(x) := \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} f(t)(x-t)^{\beta-1} dt \quad \text{and} \quad I_-^{\beta}(f)(x) := \frac{1}{\Gamma(\beta)} \int_{x}^{\infty} f(t)(t-x)^{\beta-1} dt,$$

$I_+^0(f)(x) := f(x)$, and for $-1 < \beta < 0$,

$$I_+^{\beta}(f)(x) := \frac{1}{\Gamma(1+\beta)} \frac{d}{dx} \int_{-\infty}^{x} f(t)(x-t)^{\beta} dt \quad \text{and} \quad I_-^{\beta}(f)(x) := \frac{-1}{\Gamma(1+\beta)} \frac{d}{dx} \int_{x}^{\infty} f(t)(t-x)^{\beta} dt.$$

For $H \in (0,1)$, let $\beta = H - 1/2$ and define the linear operator

$$M_{\pm}^H(f) := C_H I_{\pm}^{\beta}(f),$$

where $C_H > 0$ is chosen as in equation (1.3.3) of [31]. Then, if f is supported on $[0,\infty)$ and $M_-^H(f) \in L^2(\mathbb{R})$,

$$\int_0^{\infty} f(s)\, dW_H(s) = \int_0^{\infty} M_-^H(f)(s)\, dW_{1/2}(s),$$

see Section 1.6 of [31]. Thus, by Itô's isometry,

$$\mathbb{E}\left[\Big| \int_0^{\infty} f(s)\, dW_H(s) \Big|^2 \right] = \int_0^{\infty} M_-^H(f)(s)^2 ds = \left\| M_-^H(f) \right\|_{L^2}^2. \tag{14}$$

The following result, known as the Hardy–Littlewood inequality, provides useful upper and lower bounds of (14).

Lemma 3.1 (Corollary 1.9.2 of [31]). *There exists $L > 0$ depending on H such that the following holds. If $H \in (0, 1/2]$,*

$$\mathbb{E}\left[\Big|\int_0^\infty f(s)\,dW_H(s)\Big|^2\right] = \left\|M_-^H(f)\right\|_{L^2}^2 \geq \frac{1}{L}\,\|f\|_{L^{1/H}}^2 .$$

If $H \in [1/2, 1)$,

$$\mathbb{E}\left[\Big|\int_0^\infty f(s)\,dW_H(s)\Big|^2\right] = \left\|M_-^H(f)\right\|_{L^2}^2 \leq L\,\|f\|_{L^{1/H}}^2 .$$

3.2 A Hardy–Littlewood-type inequality for a sum of homogeneous indicator functions

Our estimates of covariances in Section 3.3 rely on a novel refinement of Lemma 3.1 for fractional integrals, which is of independent interest. It applies to a special case of step functions.

Definition 3.2. Let $k \in \mathbb{N}$. An open subset of $[0, \infty)$ is a *k-interval* if it is a union of at most k bounded open intervals.

The following theorem will later be applied with $b \in \{2, 3, \dots\}$ as in Definition 2.1. However, the theorem's statement is also true if b is not an integer.

Theorem 3.3. *Let $H \in (0, 1)$, $k \in \mathbb{N}$, $\alpha \in (0, 1)$, $b = \alpha^{-1/H}$, and $\{I_m\}_{m\in\mathbb{N}_0}$ be a collection of k-intervals with $|I_m| = b^m$. Define*

$$f_m := \alpha^m \mathbb{1}_{I_m} \quad \text{and} \quad g_M := \sum_{m=0}^{M-1} f_m. \tag{15}$$

Then there exists $L > 0$, depending on k, H, α, b, such that for all $M \in \mathbb{N}$,

$$\frac{M}{L} \leq \left\|M_-^H(g_M)\right\|_{L^2}^2 \leq LM. \tag{16}$$

Let us call a function f_m as in (15) a *homogeneous* indicator function. Theorem 3.3 then states that the L^2-norm of a fractional integral of a sum of M positive homogeneous indicator functions is of order M (up to multiplicative constants). We also remark here that if $H \in (0, 1/2)$, the classical Hardy–Littlewood inequality (Lemma 3.1) only gives the lower bound

$$\left\|M_-^H(g_M)\right\|_{L^2}^2 \geq \frac{1}{L}\,\|g_M\|_{L^{1/H}}^2 . \tag{17}$$

By Minkowski's inequality, the right-hand side of (17) further has the upper bound $\|g_M\|_{L^{1/H}}^2 \leq M^{2H}$, which for large M is strictly dominated by our bound M/L. Hence (17) cannot be optimal. A similar reasoning applies to $H \in (1/2, 1)$. Therefore, unless $H = 1/2$, our result strengthens the classical Hardy–Littlewood inequality for functions g_M.

For the proof of Theorem 3.3, we focus first on the lower bound which appears less transparent than the upper bound. The easy part of the lower bound of (16) is when $H \geq 1/2$. Indeed, since the increments of fractional Brownian motion are positively correlated for $H \geq 1/2$, we have by (15) and (14) that

$$\left\|M_-^H(g_M)\right\|_{L^2}^2 = \mathbb{E}\left[\Big|\int_0^\infty \sum_{m=0}^{M-1} f_m(s)\,dW_H(s)\Big|^2\right] \geq \sum_{m=0}^{M-1} \mathbb{E}\left[\Big|\int_0^\infty f_m(s)\,dW_H(s)\Big|^2\right].$$

Next, we write the k-interval I_m as the disjoint union of open intervals $J_1, \ldots, J_k$ (some of which may be empty). At least one of these intervals say J_1, must have length $|I_m|/k = b^m/k$. Hence, using again the positive correlation of the increments of W_H,

$$\mathbb{E}\left[\left|\int_0^\infty f_m(s)\, dW_H(s)\right|^2\right] \geq \alpha^{2m}\sum_{i=1}^{k}|J_i|^{2H} \geq \alpha^{2m}b^{2mH}k^{-2H} = k^{-2H}.$$

Hence, the lower bound in (16) holds with $L = k^{-2H}$.

Let us now turn to the case $H \in (0, 1/2)$. To lighten the notation, we define the bilinear form

$$\langle f, g\rangle := \mathbb{E}\left[\left(\int_0^\infty f dW_H\right)\left(\int_0^\infty g dW_H\right)\right]$$

on the space of continuous functions with compact support in $[0, \infty)$.

Lemma 3.4. *Let $I \subset [0, \infty)$ be a finite union of bounded open intervals, and $h : [0, \infty) \to [0, \infty)$ be a nonnegative step function (with finitely many steps), vanishing on $I^c = [0, \infty) \setminus I$. Then $\langle \mathbb{1}_I, h\rangle \geq 0$.*

Proof. We proceed by induction on the number n of disjoint open intervals in I. Consider first when I itself is an open interval. By linearity of the expectation, we may assume $h = \mathbb{1}_J$ is an indicator function of a nonempty interval $J \subseteq I$. Let us denote $I = (i_1, i_2)$ and $J = (j_1, j_2)$. Then, by the monotonicity of the function $x \mapsto x^{2H}$,

$$\langle \mathbb{1}_I, \mathbb{1}_J\rangle = \frac{1}{2}(|j_2 - i_1|^{2H} + |i_2 - j_1|^{2H} - |j_1 - i_1|^{2H} - |i_2 - j_2|^{2H}) \geq 0.$$

Next, we assume that the claim holds for n and suppose that I is the disjoint union of $n+1$ nonempty bounded open intervals, denoted by $I_1, \ldots, I_{n+1}$. We assume that $I_{n+1} = (i_{n+1}, i_{n+2})$ is the rightmost intervals so that $\sup I = i_{n+2}$. We furthermore denote $i_n := \sup I \setminus I_{n+1} \leq i_{n+1}$. By the linearity of the expectation, we may assume without loss of generality that h is the indicator function of some interval J that is contained within some I_k. By symmetry and considering the case $k = 1$, we may assume that $k \leq n$. Note then that $g := \mathbb{1}_I = g_1 + g_2$, where $g_1 = \mathbb{1}_{(i_{n+1}, i_{n+2})}$, $g_2 = \mathbb{1}_{I\setminus I_{n+1}}$. We also define

$$\widetilde{g}_1(x) := g_1(x + i_{n+1} - i_n),\ \widetilde{g} = \widetilde{g}_1 + g_2.$$

By concavity of the function $x \mapsto x^{2H}$, $\langle g_1, h\rangle \geq \langle \widetilde{g}_1, h\rangle$. This gives

$$\langle g, h\rangle = \langle g_1, h\rangle + \langle g_2, h\rangle \geq \langle \widetilde{g}_1, h\rangle + \langle g_2, h\rangle = \langle \widetilde{g}, h\rangle,$$

which is nonnegative by the induction hypothesis and noting that $\widetilde{g}$ is an indicator function of an n-interval (modulo a finite collection of points, which does not change the value of $\langle \widetilde{g}, h\rangle$). ☐

Lemma 3.5. *Let $0 < H < 1/2$ and $h : [0, \infty) \to [0, \infty)$ be a nonnegative step function whose support is I, which is a finite union of bounded open intervals. Denote by*

$$h_* := \min h(I) = \min(h([0, \infty)) \setminus \{0\}). \tag{18}$$

Then for all $\delta \leq h_*$,

$$\mathbb{E}\left[\left|\int_I h dW_H\right|^2\right] \geq \frac{\delta^2|I|^{2H}}{L} + \mathbb{E}\left[\left|\int_I (h - \delta) dW_H\right|^2\right]. \tag{19}$$

Proof. Write I as a disjoint union of bounded open intervals $I = I_1 \cup I_2 \cup \cdots \cup I_n$. Denote by h_j the restriction of h on I_j, and $\delta_j = \delta \mathbb{1}_{I_j}$, so by assumption $g_j := h_j - \delta_j \geq 0$. Thus

$$\langle h, h \rangle = \left\langle \sum_{j=1}^{N} g_j, \sum_{j=1}^{N} g_j \right\rangle + \left\langle \sum_{j=1}^{N} \delta_j, \sum_{j=1}^{N} \delta_j \right\rangle + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \langle g_i, \delta_j \rangle .$$

By Lemma 3.1 and concavity of the function $x \mapsto x^{2H}$, we have

$$\left\langle \sum_{j=1}^{N} \delta_j, \sum_{j=1}^{N} \delta_j \right\rangle \geq \frac{\delta^2}{L} \sum_{j=1}^{N} |I_j|^{2H} \geq \frac{\delta^2 |I|^{2H}}{L}.$$

Thus we conclude by Lemma 3.4 that

$$\langle h, h \rangle \geq \frac{\delta^2 |I|^{2H}}{L} + \left\langle \sum_{j=1}^{N} g_j, \sum_{j=1}^{N} g_j \right\rangle .$$

Since $\sum_{j=1}^{N} g_j = (h - \delta) \mathbb{1}_I$, we obtain (19). □

We will repeatedly apply Lemma 3.5 to bound $\mathbb{E}\left[| \int_0^\infty g_M dW_H |^2 \right]$ from below, and show that the contributions of the form $\delta^2 |I|^{2H}/L$ give the desired lower bound. One must show that h_* is large enough at some point in our procedure. This is formulated in the next result.

Lemma 3.6. *Let $M \in \mathbb{N}$ be arbitrary and recall (15). Define $S_M := g_M(\mathbb{R}) \subseteq [0, \infty)$. Then there is $L > 0$ depending only on α, k,*[2] *but not on M, m, such that for all $m < M$, $(\alpha^{m+1}, \alpha^m) \setminus S_M$ contains an open interval of length α^m / L.*

Proof. Let us choose $L_1 \in \mathbb{N}$ large with

$$\frac{3kL_1 \alpha^{L_1}}{1 - \alpha} < \frac{1 - \alpha}{2}, \tag{20}$$

which is possible since $\alpha \in (0, 1)$. Since each I_j is a k-interval, the function g_{L_1} changes value at most $2k(L_1 + 1)$ times by the definition (15), showing that $\# g_{L_1}(\mathbb{R}) \leq 2k(L_1 + 1) < 3kL_1$. Therefore, we may assume without loss of generality that $M > L_1$, because $\# g_\ell(\mathbb{R}) < 3kL_1$ for each $\ell \leq L_1$, in which case the statement can be accommodated by increasing L, once the assertion is established for $M > L_1$. For $M > L_1$, we consider a non-negative integer $m < M - L_1$ and a number $\tau \in (\alpha^{m+1}, \alpha^m) \cap S_M$. Then by (15),

$$\tau = \sum_{j=m}^{M-1} \varepsilon_j \alpha^j, \qquad \text{where } \varepsilon_j \in \{0, 1\}.$$

We define T_{m,L_1} to be the collection of all vectors $(\varepsilon_m, \ldots, \varepsilon_{m+L_1})$. Since $\# g_{L_1}(\mathbb{R}) < 3kL_1$, $\# T_{m,L_1} \leq 3kL_1$. Write $\{t_1, \ldots, t_\ell\}$ for the set of numbers that can be represented as $\sum_{j=m}^{m+L_1} \varepsilon_j \alpha^j$ for some $(\varepsilon_m, \ldots, \varepsilon_{m+L_1}) \in T_{m,L_1}$. Since $\sum_{i=L_1+1}^{\infty} \alpha^i = \alpha^{L_1+1}/(1 - \alpha)$,

$$\tau \in \bigcup_{j=1}^{\ell} \left[t_j, t_j + \sum_{i=m+L_1+1}^{\infty} \alpha^i \right] = \bigcup_{j=1}^{\ell} \left[t_j, t_j + \frac{\alpha^{m+L_1+1}}{1 - \alpha} \right]. \tag{21}$$

[2] Recall that each I_m is a k-interval.

By (20) and since $\ell \leq 3kL_1$,

$$\left|\bigcup_{j=1}^{\ell}\left[t_j, t_j+\frac{\alpha^{m+L_1+1}}{1-\alpha}\right]\right| \leq \frac{3kL_1\alpha^{m+L_1+1}}{1-\alpha} < \frac{\alpha^m(\alpha-\alpha^2)}{2},$$

which means that for $L = 6kL_1/(\alpha-\alpha^2)$ there must be an interval[3] of length α^m/L, which is a subset of

$$(\alpha^{m+1}, \alpha^m) \setminus S_M \supseteq (\alpha^{m+1}, \alpha^m) \setminus \bigcup_{j=1}^{3kL_1}\left[t_j, t_j+\frac{\alpha^{L_1+1}}{1-\alpha}\right].$$

This completes the proof for $m < M - L_1$. But since M is arbitrary and L is independent of M, the case $M - L_1 \leq m < M$ follows simply by enlarging M and proceeding with the same proof. □

Proof of Theorem 3.3. The lower bound only needs to be proved for $H < 1/2$. The idea is to repeatedly apply Lemma 3.5, "shrink" the function g_M at each step, and extract factors $\delta^2|I|^{2H}/L \geq 1/L$. Denote by L_2 the constant in Lemma 3.6. For $0 \leq i \leq M-1$, define the shrunk functions of g_M as

$$g_M^{(i)} := (g_M - \alpha^{M-i})_+,$$

which intuitively corresponds to the function $h - \delta$ in Lemma 3.5 after applying this lemma several times. Intuitively, this separates the contributions for different i using the thresholds α^{M-i-1} and α^{M-i}. By (15), clearly we have $|g_M^{-1}(\alpha^{M-i}, \infty)| \geq b^{M-i-1}$, which implies

$$|\text{supp } g_M^{(i)}| \geq b^{M-i-1}. \tag{22}$$

Let us denote the points in $S_M \cap (\alpha^{M-i-1}, \alpha^{M-i})$ by $s_M^{(i,1)} < \cdots < s_M^{(i,N_{M-i})}$, $S_M^{(i,0)} := \alpha^{M-i-1}$, $S_M^{(i,N_{M-i}+1)} := \alpha^{M-i}$, and the truncated functions $g_M^{(i,j)} := (g_M - s_M^{(i,j)})_+$. By Lemma 3.5, for all $1 \leq j \leq N_{M-i}+1$,

$$\left\langle g_M^{(i,j-1)}, g_M^{(i,j-1)} \right\rangle \geq \left\langle g_M^{(i,j)}, g_M^{(i,j)} \right\rangle + \frac{1}{L}\left(s_M^{(i,j)} - s_M^{(i,j-1)}\right)^2 |\text{supp } g_M^{(i,j-1)}|^{2H}.$$

Also by Lemma 3.6, there exists $1 \leq j_0 \leq N_{M-i}+1$ such that $s_M^{(i,j_0)} - s_M^{(i,j_0-1)} \geq \alpha^{M-i-1}/L_2$, so that by (22) and the relation $\alpha b^H = 1$,

$$\left\langle g_M^{(i)}, g_M^{(i)} \right\rangle \geq \frac{1}{L}\Big(\frac{\alpha^{M-i-1}}{L_2}\Big)^2 |\text{supp } g_M^{(i)}|^{2H} + \left\langle g_M^{(i+1)}, g_M^{(i+1)} \right\rangle \geq \frac{1}{L} + \left\langle g_M^{(i+1)}, g_M^{(i+1)} \right\rangle.$$

Summation over $0 \leq i \leq M-1$ yields that

$$\left\|M_-^H(g_M)\right\|_{L^2}^2 = \mathbb{E}\left[\Big|\int_0^\infty g_M dW_H\Big|^2\right] \geq \mathbb{E}\left[\Big|\int_0^\infty g_M^{(0)} dW_H\Big|^2\right] \geq \sum_{i=0}^{M-1}\frac{1}{L} = \frac{M}{L}.$$

This proves the lower bound.

Consider now the upper bound. Using the identity

$$\left\|M_-^H(g_M)\right\|_{L^2}^2 = \mathbb{E}\Big[\Big|\int_0^\infty g_M dW_H\Big|^2\Big] = \mathbb{E}\left[\Big|\sum_{m=0}^{M-1}\int_0^\infty f_m dW_H\Big|^2\right] = \sum_{m=0}^{M-1}\sum_{k=0}^{M-1}\langle f_m, f_k\rangle, \tag{23}$$

[3]The location of such an interval may depend on T_{L_1}, but the *existence* does not.

it suffices to bound the right-hand side of (23) from above. We use induction on M. The base case is obvious, i.e., $\langle f_0, f_0\rangle \le L$. Consider for $1 \le n \le M-1$,

$$\sum_{m=0}^{n}\sum_{k=0}^{n}\langle f_m, f_k\rangle - \sum_{m=0}^{n-1}\sum_{k=0}^{n-1}\langle f_m, f_k\rangle = \langle f_n, f_n\rangle + 2\sum_{m=0}^{n-1}\langle f_n, f_m\rangle \le L + L\sum_{m=0}^{n-1}\alpha^{m+n}\langle \mathbb{1}_{I_m}, \mathbb{1}_{I_n}\rangle .$$

If $0 < H \le 1/2$, then $\langle \mathbb{1}_{I_m}, \mathbb{1}_{I_n}\rangle \le b^{2mH}$ due to the negativity of correlations. If $H > 1/2$, by mean-value theorem, $\langle \mathbb{1}_{I_m}, \mathbb{1}_{I_n}\rangle \le Lb^m b^{n(2H-1)}$. In either case, one easily checks that

$$\sum_{m=0}^{n}\sum_{k=0}^{n}\langle f_m, f_k\rangle - \sum_{m=0}^{n-1}\sum_{k=0}^{n-1}\langle f_m, f_k\rangle \le L,$$

where L depends only on α, b. Summing over this relation completes the proof. □

3.3 Covariance estimates

In the following results, the notation $\mathbb{1}_{[c,d]}$ means $-\mathbb{1}_{[d,c]}$ if $d < c$. Recall (5) and (6).

Lemma 3.7. *Let Y be the Wiener–Weierstrass process with $H < K$, then there exists $L > 0$ such that for all $s, t \in [0,1]$ satisfying $|s-t| \le b^{-L}$,*

$$\frac{1}{L}|t-s|^{2H} \le \mathbb{E}[|Y(t)-Y(s)|^2] \le L|t-s|^{2H}.$$

In particular, Y is a quasi-helix in the sense of [22, 23].

Proof. The upper bound is straightforward by Minkowski's inequality and (5), so we focus on the lower bound. Fix $\alpha \in (0,1)$ and $b \in \{2, 3, \dots\}$ with $H \le K$. We consider a large number L_3 to be determined and we choose $L \in \mathbb{N}$ such that

$$\sum_{m=L}^{\infty}\alpha^m < \frac{1}{L_3}, \tag{24}$$

and

$$\begin{cases} \frac{\alpha^L}{\alpha b^\tau - 1} < \frac{1}{L_3} & \text{if } \alpha b^\tau > 1; \\ \forall N \ge L,\ N b^{-\tau N} < \frac{1}{L_3} & \text{if } \alpha b^\tau = 1; \\ \frac{b^{-\tau L}}{1-\alpha b^\tau} < \frac{1}{L_3} & \text{if } \alpha b^\tau < 1, \end{cases} \tag{25}$$

and for all $\delta < b^{-L}$,

$$\begin{cases} L_3\delta^{H+1} < \delta^{2H} & \text{if } \alpha b^\tau > 1; \\ L_3\left(\delta^{H+1} - \delta^2\log\delta\right) < \delta^{2H} & \text{if } \alpha b^\tau = 1; \\ L_3\left(\delta^{H+1} + \delta^2\right) < \delta^{2H} & \text{if } \alpha b^\tau < 1. \end{cases} \tag{26}$$

Fix $0 \le s \le t \le 1$ with $b^{-M-1} < t-s \le b^{-M}$ so that $M \ge L$. Observe that

$$\begin{aligned} Y(t) - Y(s) &= \sum_{m=0}^{\infty}\alpha^m(B_H(\{b^m t\}) - B_H(\{b^m s\})) \\ &= \sum_{m=0}^{\infty}\alpha^m(W_H(\{b^m t\}) - W_H(\{b^m s\}) - (\kappa(\{b^m t\}) - \kappa(\{b^m s\}))W_H(1)) = \int_0^1 g(x)\,dW_H(x) \end{aligned}$$

as a Wiener integral, where

$$g(x) := \sum_{m=0}^{\infty} \alpha^m \mathbb{1}_{[\{b^m s\},\{b^m t\}]}(x) - \sum_{m=0}^{\infty} \alpha^m(\kappa(\{b^m t\}) - \kappa(\{b^m s\})). \tag{27}$$

This integral is well-defined using Lemma 3.1 of [40]. Define

$$\ell := \inf\{1 \leq k \leq M-1 : \{b^k s\} > \{b^k t\}\} \wedge M. \tag{28}$$

We claim that for $0 \leq k < \ell$,

$$0 \leq \{b^k s\} < \{b^k t\} < 1 \text{ and } \{b^k t\} - \{b^k s\} = b^k t - b^k s \leq b^{k-M}, \tag{29}$$

and for $\ell \leq k < M$ (since $t - s < b^{-M}$),

$$0 \leq \{b^k t\} < \{b^k s\} < 1 \text{ and } \{b^k s\} - \{b^k t\} = 1 - (b^k t - b^k s) \geq 1 - b^{k-M}. \tag{30}$$

Indeed, $b^\ell t - b^\ell s \leq b^{\ell-M} < 1$ implies $\{b^\ell t\} + (1 - \{b^\ell s\}) \leq b^{\ell-M}$ and for $k \in [\ell, M)$, we have $\{b^k t\} = b^{k-\ell}\{b^\ell t\} \leq b^{-1}$ and $1 - \{b^k s\} = b^{k-\ell}(1 - \{b^\ell s\}) \leq b^{-1}$. Therefore, $\{b^k t\} < \{b^k s\}$, i.e., the order of $\{b^k t\}$ and $\{b^k s\}$ flips at most once for $0 \leq k < M$ (i.e., if $k = \ell$) and after they flip, one of them is very close to 0 and the other very close to 1, with the distances proportional to b^k.

Let us write $g(x) = \sum_{i=1}^{3} g_i(x)$ where

$$\begin{cases} g_1(x) := \sum_{m=0}^{\ell-1} \alpha^m \mathbb{1}_{[\{b^m s\},\{b^m t\}]}(x) + \sum_{m=\ell}^{M-1} \alpha^m \mathbb{1}_{[0,\{b^m t\}] \cup [\{b^m s\},1]}(x); \\ g_2(x) := -\sum_{m=0}^{\ell-1} \alpha^m(\kappa(\{b^m t\}) - \kappa(\{b^m s\})) - \sum_{m=\ell}^{M-1} \alpha^m \left(1 - (\kappa(\{b^m s\}) - \kappa(\{b^m t\}))\right); \\ g_3(x) := \sum_{m=M}^{\infty} \alpha^m \mathbb{1}_{[\{b^m s\},\{b^m t\}]}(x) - \sum_{m=M}^{\infty} \alpha^m(\kappa(\{b^m t\}) - \kappa(\{b^m s\})). \end{cases} \tag{31}$$

Note that $g_2(x)$ does not depend on x.

Let $x \in [s,t]$, then $g_1(x) \geq 1$. We also have

$$g_3(x) \geq -(1 + 2\sup|\kappa|) \sum_{m=M}^{\infty} \alpha^m =: -K_1(M).$$

By (30), for $\ell \leq k < M$, $\{b^k s\} - \{b^k t\} \geq 1 - b^{k-M}$, so that by Hölder continuity of κ,

$$\begin{aligned} g_2(x) &\geq -\sum_{m=\ell}^{M-1} \alpha^m(\{b^m t\}^\tau + (1 - \{b^m s\})^\tau) - \sum_{m=0}^{\ell-1} \alpha^m (b^m(t-s))^\tau \\ &\geq -2b^{-M\tau} \sum_{m=0}^{M-1} (\alpha b^\tau)^m \\ &\geq \begin{cases} -2(\alpha b^\tau - 1)^{-1}\alpha^M & \text{if } \alpha b^\tau > 1 \\ -2Mb^{-\tau M} & \text{if } \alpha b^\tau = 1 \\ -2(1 - \alpha b^\tau)^{-1} b^{-\tau M} & \text{if } \alpha b^\tau < 1 \end{cases} =: -K_2(M). \end{aligned}$$

Therefore by (24) and (25), and since $M \geq L$, if L_3 is large,

$$g(x) \geq 1 + g_2(x) + g_3(x) \geq 1 - (K_1(M) + K_2(M)) \geq \frac{1}{2}. \tag{32}$$

Also by (26) and since $\alpha \leq b^{-H}$, if L_3 is large,

$$K_1(M) + K_2(M) \leq \frac{1}{2H}|t-s|^{2H-1}. \tag{33}$$

Consider first the case $0 < H < 1/2$. By Lemma 3.1,

$$\mathbb{E}[|Y(t) - Y(s)|^2] \geq \frac{1}{L}\|g\|^2_{L^{1/H}} \geq \frac{1}{L}\left(\int_s^t g(x)^{1/H} dx\right)^{2H} \geq \frac{1}{L}|t-s|^{2H},$$

as required. Now let us assume that $1/2 \leq H < 1$. Expanding the square, we have

$$\begin{aligned}
\mathbb{E}(|Y(t) - Y(s)|^2) &= \mathbb{E}\left[\Big|\int_0^1 \left(g - \mathbb{1}_{[s,t]}\right) dW_H\Big|^2\right] + \mathbb{E}[|W_H(t) - W_H(s)|^2] \\
&\quad + 2\mathbb{E}\left[(W_H(t) - W_H(s))\left(\int_0^1 \left(g - \mathbb{1}_{[s,t]}\right) dW_H\right)\right] \\
&\geq \mathbb{E}[|W_H(t) - W_H(s)|^2] + 2\mathbb{E}\left[(W_H(t) - W_H(s))\left(-(K_1(M) + K_2(M))W_H(1)\right)\right] \\
&= |t-s|^{2H} - 2(K_1(M) + K_2(M))\mathbb{E}[(W_H(t) - W_H(s))W_H(1)],
\end{aligned}$$

where the second step is because W_H has non-negatively correlated increments for $H \geq 1/2$ and (32). By the mean-value theorem and since $x \mapsto x^{2H}$ is convex,

$$\mathbb{E}[(W_H(t) - W_H(s))W_H(1)] \leq H|t-s|.$$

Thus by (33),

$$\mathbb{E}[|Y(t) - Y(s)|^2] \geq |t-s|^{2H} - H|t-s|(K_1(M) + K_2(M)) \geq \frac{1}{2}|t-s|^{2H}.$$

This finishes the proof for $H \geq 1/2$. □

Observe that we only picked the interval $[s,t]$ when estimating $g_1(x)$ from below, while ignoring the other contributions from $\{[b^m s, b^m t] : m = 1, \ldots, L-1\}$. When taking care of this and other possible contributions, a more refined argument can provide a more precise estimate in the case of $H = K$.

Lemma 3.8. *Let Y be the Wiener–Weierstrass process with $H = K$, then there exists $L > 0$ such that for all $s, t \in [0,1]$ with $|s-t| \leq b^{-L}$,*

$$\frac{1}{L}|t-s|^{2H}\log\left(\frac{1}{|t-s|}\right) \leq \mathbb{E}[|Y(t) - Y(s)|^2] \leq L|t-s|^{2H}\log\left(\frac{1}{|t-s|}\right).$$

In particular, Y is a quasi-helix.

Proof. Consider a large number L and $0 \le s \le t \le 1$, $|s-t| \le b^{-L}$. Choose M such that $b^{-M-1} < t-s \le b^{-M}$. Recall from the proof of Lemma 3.7 that $Y(t) - Y(s) = \int_0^1 g(x)\, dW_H(x)$, where g is defined in (27). Define ℓ as in (28) and recall (31). Using Minkowski's inequality we will show that the contribution from g_1 is of the right order and that from g_2, g_3 are negligible. Let us consider first $1/2 \le H < 1$ so that the increments are positively correlated, which yields that

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^1 g_1 dW_H\Big|^2\Big] \\
&\ge \sum_{m=0}^{\ell-1} \mathbb{E}[\alpha^{2m}(W_H(\{b^m t\}) - W_H(\{b^m s\}))^2] + \sum_{m=\ell}^{M-1} \mathbb{E}[\alpha^{2m}(W_H(\{b^m t\}) + W_H(1) - W_H(\{b^m s\}))^2] \\
&\ge \sum_{m=0}^{\ell-1} \alpha^{2m} b^{2mH} |t-s|^{2H} + \frac{1}{L}\sum_{m=\ell}^{M-1} \alpha^{2m} b^{2mH} |t-s|^{2H} \ge \frac{1}{L}|t-s|^{2H} M \\
&\ge \frac{1}{L}|t-s|^{2H}(-\log|t-s|),
\end{aligned}$$

where the second inequality follows from the fact that

$$\begin{aligned}
\mathbb{E}[(W_H(\{b^m t\}) + W_H(1) - W_H(\{b^m s\}))^2] &\ge \max\Big\{\mathbb{E}[W_H(\{b^m t\})^2], \mathbb{E}[(W_H(1) - W_H(\{b^m s\}))^2]\Big\} \\
&\ge \left(\frac{b^m|t-s|}{2}\right)^{2H}.
\end{aligned}$$

On the other hand, to give the upper bound, we further decompose $g_1(x) = \sum_{i=1}^3 g_{1,i}(x)$ where

$$\begin{cases}
g_{1,1}(x) := \displaystyle\sum_{m=0}^{\ell-1} \alpha^m \mathbb{1}_{[\{b^m s\}, \{b^m t\}]}(x); \\
g_{1,2}(x) := \displaystyle\sum_{m=\ell}^{M-1} \alpha^m \mathbb{1}_{[0, \{b^m t\}]}(x); \\
g_{1,3}(x) := \displaystyle\sum_{m=\ell}^{M-1} \alpha^m \mathbb{1}_{[\{b^m s\}, 1]}(x).
\end{cases} \tag{34}$$

By expanding the square, the mean-value theorem, and since $x \mapsto x^{2H}$ is convex, we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^1 g_{1,1} dW_H\Big|^2\Big] \\
&= \sum_{m=0}^{\ell-1} \mathbb{E}[\alpha^{2m}(W_H(\{b^m t\}) - W_H(\{b^m s\}))^2] \\
&\qquad + 2\sum_{0 \le m < k < \ell} \mathbb{E}[\alpha^{m+k}(W_H(\{b^m t\}) - W_H(\{b^m s\}))(W_H(\{b^k t\}) - W_H(\{b^k s\}))] \\
&\le \ell|t-s|^{2H} + L\sum_{0 \le m < k < \ell} \alpha^{m+k}(b^m|t-s|)(b^k|t-s|)^{(2H-1)} \le LM|t-s|^{2H} \\
&\le L|t-s|^{2H}(-\log|t-s|).
\end{aligned}$$

Similar arguments apply for $g_{1,2}, g_{1,3}$. Therefore, by Minkowski's inequality,

$$\Big\|\int_0^1 g_1 dW_H\Big\|_2 \le \sum_{i=1}^3 \Big\|\int_0^1 g_{1,i} dW_H\Big\|_2 \le L|t-s|^H\sqrt{-\log|t-s|}.$$

Hence we conclude that

$$\frac{1}{L}|t-s|^{2H}(-\log|t-s|) \le \mathbb{E}\Big[\Big|\int_0^1 g_1 dW_H\Big|^2\Big] \le L|t-s|^{2H}(-\log|t-s|).$$

That is,

$$\frac{1}{L}|t-s|^H\sqrt{-\log|t-s|} \le \Big\|\int_0^1 g_1 dW_H\Big\|_2 \le L|t-s|^H\sqrt{-\log|t-s|}$$

We also have by Minkowski's inequality, (29), and (30),

$$\Big\|\int_0^1 g_2 dW_H\Big\|_2 = |g_2| \le |K_2(M)| \le L|t-s|^H,$$

where $K_2(M)$ is as in the proof of Lemma 3.7 and

$$\Big\|\int_0^1 g_3 dW_H\Big\|_2 \le \sum_{m=M}^{\infty} \alpha^m \Big(\big\|W_H(\{b^m t\}) - W_H(\{b^m s\})\big\|_2 + |\kappa(\{b^m t\}) - \kappa(\{b^m s\})|\Big) \le L\alpha^M = L|t-s|^H.$$

Applying again Minkowski's inequality yields that for $|t-s| < b^{-L}$ and L large enough,

$$\frac{1}{L}|t-s|^{2H}(-\log|t-s|) \le \mathbb{E}\Big[\Big|\int_0^1 g dW_H\Big|^2\Big] \le L|t-s|^{2H}(-\log|t-s|).$$

This concludes the case $H \ge 1/2$.

Now we consider the case $0 < H < 1/2$. Recall (31) and (34). Following the case $H \ge 1/2$, it suffices to show

$$\frac{1}{L}|t-s|^{2H}(-\log|t-s|) \le \mathbb{E}\Big[\Big|\int_0^1 g_1 dW_H\Big|^2\Big] \le L|t-s|^{2H}(-\log|t-s|).$$

The lower bound now follows from Theorem 3.3 (applied with $k=2$) since $b^{-M-1} < t-s \le b^{-M}$. The upper bound follows similarly as before, which we sketch below: recalling (34), we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^1 g_{1,1} dW_H\Big|^2\Big] \\
&= \sum_{m=0}^{\ell-1} \mathbb{E}[\alpha^{2m}(W_H(\{b^m t\}) - W_H(\{b^m s\}))^2] \\
&\qquad + 2\sum_{0\le m<k<\ell} \mathbb{E}[\alpha^{m+k}(W_H(\{b^m t\}) - W_H(\{b^m s\}))(W_H(\{b^k t\}) - W_H(\{b^k s\}))] \\
&\le \ell|t-s|^{2H} + L\sum_{0\le m<k<\ell} \alpha^{m+k}(b^m|t-s|)^{2H} \\
&\le L|t-s|^{2H}(-\log|t-s|),
\end{aligned}$$

and the rest follows line by line as in the case $H \ge 1/2$. □

The following result will be useful in deriving the Hausdorff dimension of the graph of fractional Wiener–Weierstrass bridges.

Lemma 3.9. *Fix $N \in \mathbb{N}$ and suppose that $K \in (2H-1, H)$. Define*

$$T_N := \{x \in [0,1] : \text{ for all } k \in \mathbb{N}_0,\ \{b^k x\} \in [b^{-N}, 1-b^{-N}]\}. \tag{35}$$

Then for all $t, s \in T_N$ with $|t-s| < b^{-L}$,

$$\mathbb{E}[|Y(t)-Y(s)|^2] \geq \frac{1}{L}|t-s|^{2K}.$$

Here, L may depend on N.

Proof. We fix $t, s \in T_N$ and $M \in \mathbb{N}_0$ with $b^{-M-1} < t-s \leq b^{-M}$, where $M > N$ is large and will be determined later. The central fact used here is $\{b^m s\} < \{b^m t\}$ for $0 \leq m < M-N$, because otherwise $\{b^m t\} + 1 - \{b^m s\} = |t-s|b^m \leq b^{m-M} < b^{-N}$, contradicting $\{b^m s\}, \{b^m t\} \in [b^{-N}, 1-b^{-N}]$. Note also that for $0 \leq m < M-N$, $\{b^m t\} - \{b^m s\} = |t-s|b^m$.

Recall from the proof of Lemma 3.7 that $Y(t) - Y(s) = \int_0^1 g(x)\, dW_H(x)$ where

$$g(x) := \sum_{m=0}^{\infty} \alpha^m \mathbb{1}_{[\{b^m s\}, \{b^m t\}]}(x) - \sum_{m=0}^{\infty} \alpha^m (\kappa(\{b^m t\}) - \kappa(\{b^m s\})).$$

First,

$$\left| \sum_{m=0}^{\infty} \alpha^m (\kappa(\{b^m t\}) - \kappa(\{b^m s\})) \right| \leq L \sum_{m=0}^{M-N-1} \alpha^m (b^m |t-s|)^H + L \sum_{m=M-N}^{\infty} \alpha^m \leq L|t-s|^{-\log_b(\alpha)}.$$

Similarly,

$$\left| \sum_{m=M-N}^{\infty} \alpha^m \mathbb{1}_{[\{b^m s\}, \{b^m t\}]}(x) \right| \leq L|t-s|^{-\log_b(\alpha)}.$$

Here, L depends on κ, α, and N, but not on M. Thus there exists $L_4 > N$ such that

$$2\left| -\sum_{m=0}^{\infty} \alpha^m (\kappa(\{b^m t\}) - \kappa(\{b^m s\})) + \sum_{m=M-N}^{\infty} \alpha^m \mathbb{1}_{[\{b^m s\}, \{b^m t\}]}(x) \right| \leq \alpha^{M-L_4}.$$

Since L_4 does not depend on M, we may choose M so that $M > L_4$. Then, for $0 \leq m \leq M - L_4$ and $x \in [\{b^m s\}, \{b^m t\}]$,

$$g(x) \geq \sum_{k=0}^{M-L_4} \alpha^k \mathbb{1}_{[\{b^k s\}, \{b^k t\}]}(x) - \frac{1}{2}\alpha^m \geq \frac{1}{2}\alpha^m.$$

Let us consider first the case $0 < H < 1/2$. By Lemma 3.1 and our previous observation that $\{b^{M-L_4} t\} - \{b^{M-L_4} s\} = b^{M-L_4}|t-s|$,

$$\mathbb{E}[|Y(t)-Y(s)|^2] \geq \frac{1}{L}\left(\int_0^1 g^{1/H}\, dx\right)^{2H} \geq \frac{1}{L}\left(b^{M-L_4}|t-s| \left(\frac{\alpha^m}{2}\right)^{1/H}\right)^{2H} \geq \frac{1}{L}|t-s|^{-2\log_b(\alpha)}.$$

Now we consider the case $1/2 \leq H < 1$, where we suppose that $\alpha b^{2H-1} < 1$. Consider a large number $L_5 > L_4$ to be determined, and let

$$h(x) := g(x) - \frac{1}{2}\alpha^{M-L_5} \mathbb{1}_{[\{b^{M-L_5} s\}, \{b^{M-L_5} t\}]}(x).$$

Therefore, by expanding the square,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^1 g dW_H\Big)^2\Big] \\
&\geq \mathbb{E}\Big[\Big(\int_0^1 \frac{1}{2}\alpha^{M-L_5}\mathbb{1}_{[\{b^{M-L_5}s\},\{b^{M-L_5}t\}]}dW_H\Big)^2\Big] \\
&\qquad\qquad - \mathbb{E}\Big[\Big(\int_0^1 \alpha^{M-L_5}\mathbb{1}_{[\{b^{M-L_5}s\},\{b^{M-L_5}t\}]}dW_H\Big)\Big(\int_0^1 h dW_H\Big)\Big] \\
&\geq \frac{1}{L}\alpha^{2M-2L_5}b^{2(M-L_5)H}|t-s|^{2H} - \alpha^{M-L_5}\sup|h|\mathbb{E}[W_H(1)(W_H(\{b^{M-L_5}t\}) - W_H(\{b^{M-L_5}s\}))] \\
&\geq \frac{1}{L}\alpha^{2M-2L_5}b^{-2L_5H} - L\alpha^{2M-L_5}b^{-L_5},
\end{aligned}
$$

where in the second inequality we used that the increments of W_H are positively correlated. Since $\alpha b^{2H-1} < 1$, for L_5 large enough we have $\alpha^{-2L_5}b^{-2L_5H} > L\alpha^{-L_5}b^{-L_5}$, which yields that

$$
\mathbb{E}\Big[\Big(\int_0^1 g dW_H\Big)^2\Big] \geq \frac{1}{L}\alpha^{2M} = \frac{1}{L}|t-s|^{2K}.
$$

This finishes the proof. □

4 Proofs of the main results

In this section, we prove Theorem 2.2 in Section 4.1, Theorems 2.4, 2.3, and 2.6 in Section 4.2, and finally Theorem 2.7 in Section 4.3.

4.1 Φ-variation

We first prove a general result for the Φ-variation of Gaussian processes, extending Theorem 4 of [24] to processes with non-stationary increments. A corresponding but informal discussion was initiated in Section 10.6 of [29], while no proofs or precise statements were given. Here, we provide a formal theorem with a detailed proof that requires weaker conditions on the covariance structure.

Theorem 4.1. *Consider a centered Gaussian process $(X(t))_{t\in[0,1]}$ with*

$$
\frac{1}{L}\sigma(|h|) \leq \big\|X(t+h) - X(t)\big\|_2 \leq L\sigma(|h|) \tag{36}
$$

for all $t, t+h \in [0,1]$, where σ is concave and regularly varying at 0 with index $H \in (0,1)$. Suppose $\Psi(x) := \sigma(x)(\log\log(1/x))^{1/2}$ is strictly increasing and denote by Φ its inverse, then

$$
\mathbb{P}\left(\frac{1}{L} < v_\Phi(X) < \infty\right) = 1
$$

for this choice of Φ.

We first apply the result to prove Theorem 2.2. In reality, it is difficult to write down explicit formulas of Φ given σ, but the essential point of $\Phi = \Psi^{-1}$ is to make the following equation (37) hold. By Proposition 1.5.15 of [6], Φ is regularly varying with index $1/H$ at 0. Hence, for any $c > 0$,

$$
\lim_{v\to 0}\frac{\Phi(c\Psi(v))}{v} = \lim_{v\to 0}\frac{\Phi(c\Psi(v))}{\Phi(\Psi(v))} = c^{1/H}. \tag{37}
$$

In the proof of Theorem 4.1, we will only apply $\Phi = \Psi^{-1}$ indirectly through (37).

Proof of Theorem 2.2. Let us first consider the case $H > K$. By Proposition A.1 of [40], the sample paths of Y are Hölder with exponent K, and so $v_\Phi(Y) < \infty$ a.s. On the other hand, Theorem 2.3(a) of [40] implies that $v_\Phi(Y) > 0$ a.s. This proves (iii) of Theorem 2.2.

Now we turn to parts (i) and (ii). Given the covariance estimates in Section 3.3 and by chopping $[0, 1]$ into intervals of length $< b^{-L}$ (because the estimates in Lemma 3.7 and Lemma 3.8 apply only for $|s - t| < b^{-L}$), it suffices to prove (37) in the cases $H \le K$ with our choices of Φ from (8) and (9). For $H < K$ we may refer to (12.42) of [9]. The case $H = K$ is settled by a direct computation:

$$\begin{aligned}
&\lim_{v\to 0} \frac{\Phi_H(c\Psi_H(v))}{v} \\
&= \lim_{v\to 0} \left(\frac{c\sqrt{2v^{2H}\log(1/v)\log\log(1/v)}}{\sqrt{-2\log(c\sqrt{2v^{2H}\log(1/v)\log\log(1/v)})\log(-\log(c\sqrt{2v^{2H}\log(1/v)\log\log(1/v)}))/H}} \right)^{1/H} v^{-1} \\
&= c^{1/H} \lim_{v\to 0} \left(\frac{2\log(1/v)\log\log(1/v)}{-2\log(c\sqrt{2v^{2H}\log(1/v)\log\log(1/v)})\log(-\log(c\sqrt{2v^{2H}\log(1/v)\log\log(1/v)}))/H} \right)^{1/2H} \\
&= c^{1/H} \lim_{v\to 0} \left(\frac{-2\log(v)\log(-\log v)}{-2\log(v^H)\log(-\log(v^H))/H} \right)^{1/2H} \\
&= c^{1/H}.
\end{aligned}$$

This completes the proof. □

Theorem 4.1 is a consequence of Corollary 4.3 below, by following the proof of Corollary 12.23 in [9]. For a partition $\kappa = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0, 1]$, we denote its mesh by $|\kappa|$ and define

$$s_\Phi(f, \kappa) := \sum_{i=1}^{n} \Phi(|f(t_i) - f(t_{i-1})|).$$

In the following, consider a Gaussian process X satisfying the conditions in Theorem 4.1.

Theorem 4.2. *Under the above conditions,*

$$\mathbb{P}\left(\frac{1}{L} \le \lim_{\delta\to 0} \sup\{s_\Phi(X, \kappa) : |\kappa| < \delta\} \le L \right) = 1.$$

Corollary 4.3. *There exists a constant* $C \in (0, \infty)$ *such that*

$$\mathbb{P}\left(\lim_{\delta\to 0} \sup\{s_\Phi(X, \kappa) : |\kappa| < \delta\} = C \right) = 1.$$

Proof. This follows from Theorem 1 of [24]. □

The proof of Theorem 4.2 will follow the same path as Theorem 12.22 of [9]. Let us first present some preparatory lemmas. In the statement of these lemmas, we impose the same assumptions as Theorem 4.1.

Lemma 4.4. *For all* $y > 1$ *and all* $0 \le h < h + \delta \le 1$,

$$\mathbb{P}\Bigg(\sup_{s,t\in[h,h+\delta]} |X(s) - X(t)| > Ly\sigma(\delta) \Bigg) \le L(1 + L^{-1})^{-y^2}.$$

Proof. Let us first compute $\mathbb{E}\big[\sup_{t\in[h,h+\delta]} X(t)\big]$. Since Ψ is regularly varying, there is $\alpha > 0$ such that for k large enough, $\Psi(2^{-k-1}) \le 2^{-\alpha}\Psi(2^{-k})$. If $\delta \in [2^{-N}, 2^{-N+1})$, we have for N large enough, since $\Psi(x) = \sigma(x)\sqrt{\log\log(1/x)}$ is non-decreasing,

$$\sum_{n=0}^{\infty} 2^{n/2}\left(\frac{\sigma(\delta 2^{-2^n})}{\sigma(\delta)}\right) \le \sum_{n=0}^{\infty} 2^{n/2}\left(\frac{\Psi(2^{-N+1-2^n})\sqrt{\log N}}{\Psi(2^{-N})\sqrt{\log(N+2^n-1)}}\right) \le \sum_{n=0}^{\infty} 2^{n/2}2^{\alpha(1-2^n)} \le L.$$

By (36) and Dudley's entropy bound (Proposition 2.5.1 of [42]),

$$\mathbb{E}\Big[\sup_{t\in[h,h+\delta]} X(t)\Big] \le L\sum_{n=0}^{\infty} 2^{n/2}\sigma(\delta 2^{-2^n}) \le L\sigma(\delta).$$

Lemma 2.2.1 of [42] and the fact that X is symmetric yield that

$$\mathbb{E}\Big[\sup_{s,t\in[h,h+\delta]} |X(s)-X(t)|\Big] \le L\sigma(\delta).$$

Thus by the Gaussian concentration inequality (Theorem 5.4.3 and Corollary 5.4.5 of [29] applied to a dense subset of $[h, h+\delta]$), since $y > 1$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{s,t\in[h,h+\delta]}(X(s)-X(t)) > Ly\sigma(\delta)\Big)\\
&\le \mathbb{P}\Big(\Big|\sup_{s,t\in[h,h+\delta]}(X(s)-X(t)) - \mathbb{E}\Big[\sup_{s,t\in[h,h+\delta]}(X(s)-X(t))\Big]\Big| > \frac{1}{2}Ly\sigma(\delta)\Big)\\
&\le L\exp\left(\frac{-(\frac{1}{2}Ly\sigma(\delta))^2}{L\sup_{s,t\in[h,h+\delta]}\mathbb{E}[|X(t)-X(s)|^2]}\right)\\
&\le L(1+L^{-1})^{-y^2},
\end{aligned}$$

as required, where the last step follows from (36). This proves the lemma while noting that the absolute value on $X(s)-X(t)$ can be removed by symmetry. ◻

Lemma 4.5. *There exist constants $C(s) \in [\frac{1}{L}, L]$, $s \in [0,1]$ such that*

$$\mathbb{P}\left(\limsup_{t\in[0,1],\, t\to s} \frac{|X(t)-X(s)|}{\Psi(|t-s|)} = C(s)\right) = 1. \tag{38}$$

In addition,

$$\mathbb{P}\left(\limsup_{\delta\to 0}\sup_{|t-s|<\delta} \frac{|X(t)-X(s)|}{\widetilde{\Psi}(|t-s|)} \le L\right) = 1, \tag{39}$$

where $\widetilde{\Psi}(x) = \sigma(x)\sqrt{-\log x}$.

Proof. The proof of (38) is essentially the same as that of Theorem 2.3 below and hence omitted. Note that $C(s) \in [\frac{1}{L}, L]$, otherwise it would contradict Lemma 7.1.10 of [29] with the choice $Y(t) = L^{\pm 2}X(t+(s'-s))$, where L is as in (36). The claim (39) follows from (36), Theorem 7.2.1 and Lemma 7.2.5 in [29]. ◻

Proof of Theorem 4.2. Consider first the lower bound. Denote by L_6 the constant in Lemma 4.5 so that $C(s) \geq 1/L_6$ for all s. Let $\varepsilon \in (0,1)$ be arbitrary,[4] and

$$E(\delta) := \left\{ (t,\omega) : \ \exists s \in (0,\delta) \cap \mathbb{Q}, \ \Phi(|X(t+s,\omega) - X(t,\omega)|) > \frac{(1-\varepsilon)s}{L_6^{1/H}} \right\}. \tag{40}$$

Taking $c = L_6^{-1}$ in (37) yields that there is δ such that for all $0 < s < \delta$,

$$\Phi\left(\frac{\Psi(s)}{L_6}\right) \geq \frac{(1-\varepsilon)s}{L_6^{1/H}}.$$

Since Φ is increasing, (38) yields that for each $t \in (0,1)$ the t-section $E_t(\delta)$ of $E(\delta)$ is such that $\mathbb{P}(E_t(\delta)) = 1$. It follows from Fubini's theorem that $\mathbb{P}(|E_\delta| = 1) = 1$. Observe that the set of intervals $[t, t+s]$ with $t \in [0,1]$ and s as in (40) form a Vitali covering of

$$E := \bigcap_{0<\delta\leq 1} E(\delta) = \bigcap_{k=1}^{\infty} E(1/k).$$

By Vitali's covering theorem (e.g., Theorem 1 in §8 of Chapter III in [33]), we may choose a finite subcollection of disjoint intervals $\{[t'_j, t'_j + s_j]\}$ with a total length of at least $1-\varepsilon$. Then for a partition κ with mesh $|\kappa| < \delta$,

$$s_\Phi(X,\kappa) \geq \sum_j \Phi(|X(t'_j + s_j, \omega) - X(t'_j,\omega)|) \geq \frac{1-\varepsilon}{L_6^{1/H}} \sum_j s_j \geq \frac{(1-\varepsilon)^2}{L_6^{1/H}} \geq \frac{1}{L}.$$

Let us now focus on the upper bound. For any partition $\kappa = \{t_i\}_{i=0}^n$ of $[0,T]$, let $\Delta_i = t_i - t_{i-1}$ and $\Delta_i X := X(t_i) - X(t_{i-1})$ for $1 \leq i \leq n$. Let I_1, I_2 be sets of $i \in \{1,\ldots,n\}$ such that $|\Delta_i X|$ is respectively in $[0, L_7\Psi(\Delta_i)]$ and $[L_7\Psi(\Delta_i), \infty)$. The constant L_7 will be determined later. For any $c, \varepsilon > 0$ there is $\eta(c,\varepsilon) > 0$ such that for all $v \in (0, \eta(c,\varepsilon))$, $\Phi(c\Psi(v)) \leq (c^{1/H} + \varepsilon)v$. Choose $\delta_1 = \eta(L_7 + \varepsilon, \varepsilon)$, so for any partition κ with $|\kappa| < \delta_1$,

$$\sum_{i\in I_1} \Phi(|\Delta_i X|) \leq \sum_{i\in I_1} \Phi(L_7\Psi(\Delta_i)) \leq (L_7+\varepsilon)^{1/H} \sum_{i\in I_1} \Delta_i \leq L. \tag{41}$$

To estimate the sum over I_2, let $S_{m,j} = [(j/2)e^{-m}, (1+j/2)e^{-m}]$ and

$$V_{m,j} := \left\{ \omega \in \Omega : \sup_{s,t\in S_{m,j}} |X(t,\omega) - X(s,\omega)| \geq L_7\Psi(e^{-m-2}) \right\}.$$

Let $j_m = \lfloor 2e^m \rfloor - 1$. We have

$$\#\Lambda_m := \#\{i \in I_2 : e^{-m-1} < 2\Delta_i \leq e^{-m}\} \leq 5\#\{0 \leq j \leq j_m : \omega \in V_{m,j}\} =: Z_m(\omega).$$

Denote by L_8 the constant in Lemma 4.4. Take L_9 with $(1 + L_8^{-1})^{L_9^2 e^{-8H}} \geq e^{4+2/H}$ and $L_7 = L_8 L_9$, we obtain by Lemma 4.4 with $\delta = e^{-m}$ and $y = L_9\Psi(e^{-m-2})/\sigma(e^{-m})$ that for m large enough,

$$\mathbb{P}(V_{m,j}) \leq L_8(1 + L_8^{-1})^{-y^2} \leq L(m+2)^{-4-2/H},$$

[4] In fact, one can just take $\varepsilon = 1/2$ everywhere.

thus $\mathbb{E}[Z_m] \le Le^m m^{-4-2/H}$, so that by Markov's inequality and the Borel–Cantelli lemma, with probability one there is $m_1 = m_1(\omega) > 3$ such that for all $m > m_1(\omega)$, $Z_m(\omega) \le e^m m^{-2-2/H}$.

Since σ is regularly varying with index H, using Karamata's representation we write

$$\sigma(x) = x^H \beta(x) \exp\left(\int_1^x \frac{\alpha(u)}{u} du\right),$$

where $\alpha(u) \to 0, \beta(x) \to C \neq 0$ as $x \to 0$. It is then easy to see that

$$\frac{\Psi(e^{-m}m^{2/H})}{\Psi(e^{-m})} \ge \frac{\sigma(e^{-m}m^{2/H})}{\sigma(e^{-m})} \ge m \tag{42}$$

for m large enough, say $m > m_0$.

By (39), there exists $K(\omega) < \infty$ such that $|X(s) - X(t)| \le K(\omega)\widetilde{\Psi}(|t-s|)$ for all $s, t \in [0,1]$ and for almost every ω. Pick $m_2(\omega) = \max\{m_0, m_1(\omega)\}$ and define $\delta_2 = \delta_2(\omega) = e^{-m_2(\omega)} \wedge \eta(K(\omega), \varepsilon)$. Then for κ with $|\kappa| < \delta_2$, each $m \ge m_2(\omega)$, and each $i \in \Lambda_m$, there is j such that $[t_{i-1}, t_i] \subseteq S_{m,j}$. Thus by (42) applied on the second line,

$$\begin{aligned}
\sum_{i \in I_2} \Phi(|\Delta_i X|) &= \sum_{m \ge m_2(\omega)} \sum_{i \in I_2 \cap \Lambda_m} \Phi(|\Delta_i X|) \\
&\le \sum_{m \ge m_2(\omega)} Z_m(\omega) \Phi(K(\omega)\widetilde{\Psi}(e^{-m})) \\
&\le \sum_{m \ge m_2(\omega)} e^m m^{-2-2/H} \Phi(K(\omega)\Psi(e^{-m}m^{2/H})) \\
&\le \sum_{m \ge m_2(\omega)} m^{-2}(K(\omega)^{1/H} + \varepsilon) \le L.
\end{aligned} \tag{43}$$

Combining (41) and (43) yields that with probability one, for any partition κ with $|\kappa| < \delta_1 \wedge \delta_2$,

$$s_\Phi(X, \kappa) \le \sum_{j=1}^{2} \sum_{i \in I_j} \Phi(|\Delta_i X|) \le L,$$

completing the proof of the upper bound. Lastly, we note that the final assertion involving the function Θ is obvious. □

Finally, we remark that the upper bound part of Theorem 4.1 may also follow from Theorem 1.3 of [3] along with regular variation techniques in [6]. On the other hand, Theorem 4.1 is stronger in the sense that it characterizes the critical Φ for which the Φ-variation is non-trivial.

4.2 Moduli of continuity and Hausdorff dimension

We will frequently use the following zero-one law on the moduli of continuity for Gaussian processes.

Lemma 4.6 (Lemma 7.1.1 of [29]). *Let $(G(t))_{t \in [0,1]}$ be a centered Gaussian process for which $d(t,s) := \mathbb{E}[(G(t) - G(s))^2]$ is continuous. Let furthermore $\omega, \rho : [0,1) \to [0,\infty)$ be continuous functions with $\omega(0) = \rho(0) = 0$, and $K \in [0,1]$ be a compact set. Then*

$$\lim_{h \to 0} \sup_{\substack{t,s \in K \\ |t-s|<h}} \frac{G(t) - G(s)}{\omega(|t-s|)} \le L \quad a.s. \quad \Longrightarrow \quad \lim_{h \to 0} \sup_{\substack{t,s \in K \\ |t-s|<h}} \frac{G(t) - G(s)}{\omega(|t-s|)} = L' \quad a.s.$$

for some $L' \geq 0$. Similarly, for each $s \in [0,1]$,

$$\limsup_{t\in[0,1],\, t\to s} \frac{G(t)-G(s)}{\rho(|t-s|)} \leq L_s \quad a.s. \quad \Longrightarrow \quad \limsup_{t\in[0,1],\, t\to s} \frac{G(t)-G(s)}{\rho(|t-s|)} = L'_s \quad a.s.$$

for some $L'_s \geq 0$.

Proof of Theorem 2.3. (i) This follows from Lemma 3.7 and Theorem 7.6.4 of [29], applied with $\phi(h) = h^{2H}$.

(ii) This follows from Lemma 3.8 and Theorem 7.6.4 of [29], applied with $\phi(h) = h^{2H}(-\log h)$.

(iii) By pathwise Hölder continuity (Proposition A.1 of [40]), for each $s \in [0,1]$,

$$\limsup_{t\in[0,1],\, t\to s} \frac{|Y(t)-Y(s)|}{|t-s|^K} < \infty \quad \text{a.s.},$$

and hence the random variable Z_s is well-defined. It remains to show that Z_s is strictly positive non-constant, and unbounded for almost every $s \in [0,1]$. We fix a large integer $L_{10} > 0$ to be determined, and consider the set of $s \in [0,1]$ such that there exists an infinite sequence $n_k \to \infty$ such that for all k, $\{sb^{n_k - L_{10}}\}, \{sb^{n_k - L_{10}+1}\}, \ldots, \{sb^{n_k}\} \in [0, 1/3]$. This condition holds for almost every $s \in [0,1]$ by considering the b-adic decimal expansion.

Let

$$X_n = \alpha^{-n}\big(Y(b^{-n}+s) - Y(s)\big).$$

We claim that

$$\limsup_{k\to\infty} \mathbb{E}[|X_{n_k}|^2] > 0. \tag{44}$$

Suppose that (44) holds. Since each X_{n_k} is Gaussian, for each $x \geq 0$, there exists $\delta_x > 0$ such that

$$\mathbb{P}\Big(\limsup_{t\in[0,1],\, t\to s} \frac{|Y(t)-Y(s)|}{|t-s|^K} > x\Big) \geq \mathbb{P}\Big(\limsup_{k\to\infty} |X_{n_k}| > x\Big) \geq \limsup_{k\to\infty} \mathbb{P}(|X_{n_k}| > x) \geq \delta_x > 0,$$

where the second inequality follows from Fatou's lemma. Taking $x = 0$ yields $Z_s > 0$ a.s., and taking $x \to \infty$ yields that Z_s is non-constant and unbounded, as desired.

It remains to establish (44). Using (5), we write

$$X_n = \sum_{m=0}^{\infty} \alpha^{m-n}\big(B_H(\{b^{m-n}+b^m s\}) - B_H(\{b^m s\})\big) = \sum_{m=1}^{n} \alpha^{-m}\big(B_H(\{b^{-m}+b^{n-m}s\}) - B_H(\{b^{n-m}s\})\big).$$

By Minkowski's inequality and since $\alpha b^H > 1$, we have

$$\Big\| \sum_{m=L_{10}+1}^{n} \alpha^{-m}\big(B_H(\{b^{-m}+b^{n-m}s\}) - B_H(\{b^{n-m}s\})\big)\Big\|_2 \leq \sum_{m=L_{10}+1}^{n} \alpha^{-m} b^{-mH} \leq L(\alpha b^H)^{-L_{10}}. \tag{45}$$

On the other hand, for $n = n_k$, we write

$$
\begin{aligned}
&\sum_{m=1}^{L_{10}} \alpha^{-m}\big(B_H(\{b^{-m} + b^{n_k-m}s\}) - B_H(\{b^{n_k-m}s\})\big) \\
&= \sum_{m=1}^{L_{10}} \alpha^{-m}\big(B_H(b^{-m} + \{b^{n_k-m}s\}) - B_H(\{b^{n_k-m}s\})\big) \\
&= \sum_{m=1}^{L_{10}} \alpha^{-m}\big(W_H(b^{-m} + \{b^{n_k-m}s\}) - W_H(\{b^{n_k-m}s\})\big) \\
&\qquad\qquad - \sum_{m=1}^{L_{10}} \alpha^{-m}\big(\kappa(b^{-m} + \{b^{n_k-m}s\}) - \kappa(\{b^{n_k-m}s\})\big)W_H(1) \\
&=: \int_0^1 \left(\psi_{n_k}(t) - \sum_{m=1}^{L_{10}} \alpha^{-m}\big(\kappa(b^{-m} + \{b^{n_k-m}s\}) - \kappa(\{b^{n_k-m}s\})\big)\right) dW_H(t).
\end{aligned}
$$

Since κ is strictly increasing, there is a constant $\delta > 0$ independent of L_{10} such that

$$
\sum_{m=1}^{L_{10}} \alpha^{-m}\big(\kappa(b^{-m} + \{b^{n_k-m}s\}) - \kappa(\{b^{n_k-m}s\})\big) > \delta.
$$

Moreover, by construction, each ψ_{n_k} is supported on $[0, 5/6]$. By the strong local non-determinism of W_H, we have

$$
\left\| \int_0^1 \left(\psi_{n_k}(t) - \sum_{m=1}^{L_{10}} \alpha^{-m}\big(\kappa(b^{-m} + \{b^{n_k-m}s\}) - \kappa(\{b^{n_k-m}s\})\big)\right) dW_H(t) \right\|_2 \geq \delta
$$

for some $\delta > 0$. Altogether, by Minkowski's inequality and (45), we conclude that

$$
\|X_{n_k}\|_2 \geq \delta - L((\alpha b^H)^{-L_{10}} + \alpha^{L_{10}}).
$$

Since δ, L do not depend on L_{10}, picking L_{10} large enough yields (44), completing the proof. □

Suppose that $H \leq K$. In this case, Theorem 2.3 is an immediate consequence of covariance estimates from Section 3.3. On the other hand, covariance estimates do not suffice for establishing Theorem 2.4. The following new strategy will be needed: we consider a large n and compare the Wiener–Weierstrass process Y as defined in (5) with a new process

$$
X^{(n)}(t) := \sum_{m=0}^{n-1} \alpha^m W_H(\{b^m t\}). \tag{46}
$$

One easily sees that by the uniform modulus of continuity of W_H, the sum converges as $n \to \infty$ if $H < K$. Hence, when studying the limit behavior of the process Y, it suffices to consider $X^{(n)}$ because their difference is of a small scale. A few changes will be needed in the critical case $H = K$ where one can only truncate the series but cannot replace the bridge B_H by the fractional Brownian motion W_H.

The following technical lemma establishes the uniform modulus of continuity of $X^{(n)}$. Note that when approximating Y with $X^{(n)}$, the number n depends on the scale we look at, which motivates the choice of N_n below.

Lemma 4.7. *If $H < K$ we define $X^{(n)}$ as in (46), and $\varepsilon_n = \alpha^n$, $N_n = nH$, $\delta_n = b^{-nH}$.*[5] *Then we have*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{\substack{t,s\in[0,1]\\ |t-s|<\varepsilon_n/\delta_n}} \left|X^{(N_n)}(t) - X^{(N_n)}(s)\right| \le \frac{1}{L}\left(\frac{\varepsilon_n}{\delta_n}\right)^H \sqrt{-\log(\varepsilon_n/\delta_n)} \right) < \infty. \tag{47}$$

Proof. The plan is to apply the Sudakov minoration (e.g. Lemma 2.10.2 of [42]) to the increments of $X^{(N_n)}$ on well-spaced subintervals of $[0,1]$. To achieve this goal, we first need to estimate the covariances. We consider two large constants L_{11}, L_{12} to be determined later, and for each n consider subintervals of length $\varepsilon_n/(L_{11}\delta_n)$ of $[1/b, 2/3]$ that are at least $L_{12}\varepsilon_n/(2L_{11}\delta_n)$ apart. We consider the collection of intervals

$$\left\{ \left[\frac{1}{b} + \frac{jL_{12}\varepsilon_n}{L_{11}\delta_n}, \frac{1}{b} + \frac{(jL_{12}+1)\varepsilon_n}{L_{11}\delta_n} \right] \right\}_{1\le j\le j_n},$$

where j_n is as large as possible such that the intervals lie in $[1/b, 2/3]$. One easily sees that $j_n \ge \delta_n/(L\varepsilon_n)$. Let us define for $1 \le j \le j_n$,

$$\Delta^X_{n,j} := X^{(N_n)}\left(\frac{1}{b} + \frac{(jL_{12}+1)\varepsilon_n}{L_{11}\delta_n}\right) - X^{(N_n)}\left(\frac{1}{b} + \frac{jL_{12}\varepsilon_n}{L_{11}\delta_n}\right),$$

$$\Delta^W_{n,m,j} := W_H\left(b^m\left(\frac{1}{b} + \frac{(jL_{12}+1)\varepsilon_n}{L_{11}\delta_n}\right)\right) - W_H\left(b^m\left(\frac{1}{b} + \frac{jL_{12}\varepsilon_n}{L_{11}\delta_n}\right)\right).$$

We then have for $1 \le k < \ell \le j_n$, for some measurable function f_n,

$$\begin{aligned}
&\mathbb{E}\left[\left|\Delta^X_{n,k} - \Delta^X_{n,\ell}\right|^2\right] \\
&= \mathbb{E}\left[\left|\sum_{m=0}^{N_n-1} \alpha^m(\Delta^W_{n,m,k} - \Delta^W_{n,m,\ell})\right|^2\right] \\
&= \mathbb{E}\left[\left|W_H\left(\frac{1}{b} + \frac{kL_{12}\varepsilon_n}{L_{11}\delta_n}\right) - W_H\left(\frac{1}{b} + \frac{(kL_{12}+1)\varepsilon_n}{L_{11}\delta_n}\right) + \int_{\frac{1}{b}+\frac{(kL_{12}+1)\varepsilon_n}{L_{11}\delta_n}}^{\infty} f_n(x)\, dW_H(x)\right|^2\right] \\
&\ge \frac{1}{L}\left(\frac{\varepsilon_n}{L_{11}\delta_n}\right)^{2H},
\end{aligned}$$

where the last step follows from the strong local non-determinism applied to W_H (Lemma 7.1 of [36]). Thus by the Sudakov minoration (e.g., Lemma 2.10.2 of [42]),

$$\mathbb{E}\left[\sup_{1\le j\le j_n} \Delta^X_{n,j}\right] \ge \frac{1}{L}\left(\frac{L_{12}\varepsilon_n}{L_{11}\delta_n}\right)^{2H} \log(j_n) \ge \frac{1}{L}\left(\frac{\varepsilon_n}{\delta_n}\right)^H \sqrt{-\log(\varepsilon_n/\delta_n)}.$$

Moreover, for any j,

$$\left\|\Delta^X_{n,j}\right\|_2^2 \le L\left(\frac{\varepsilon_n}{\delta_n}\right)^{2H},$$

which follows from the negativity of covariances if $0 < H < 1/2$ and Lemma 3.1 if $1/2 \le H < 1$. By the Gaussian concentration inequality (Theorem 5.4.3 and Corollary 5.4.5 of [29]),

$$\mathbb{P}\left(\left|\sup_{1\le j\le j_n} \Delta^X_{n,j} - \mathbb{E}\left[\sup_{1\le j\le j_n} \Delta^X_{n,j}\right]\right| \ge u\right) \le L\exp\left(\frac{-u^2}{L(\varepsilon_n/\delta_n)^{2H}}\right).$$

[5] Without loss of generality, we assume N_n is always an integer, by using instead $\lfloor nH \rfloor$ or $\lceil nH \rceil$.

Taking $u = L(\varepsilon_n/\delta_n)^H(-\log(\varepsilon_n/\delta_n))^{1/4} \ll (\varepsilon_n/\delta_n)^H\sqrt{-\log(\varepsilon_n/\delta_n)}$ shows that

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sup_{1\le j\le j_n}\Delta^X_{n,j} \le \frac{1}{L}(\frac{\varepsilon_n}{\delta_n})^H\sqrt{-\log(\varepsilon_n/\delta_n)}\right) < \infty.$$

Then (2) holds from the trivial bound

$$\sup_{\substack{t,s\in[0,1]\\ |t-s|<\varepsilon_n/\delta_n}}\left(X^{(N_n)}(t) - X^{(N_n)}(s)\right) \ge \sup_{1\le j\le j_n}\Delta^X_{n,j}.$$

This finishes the proof. □

Proof of Theorem 2.4. (i) Consider the sequence of numbers $\varepsilon_n, \delta_n, N_n$ as in Lemma 4.7 and denote by $\phi(n) = \varepsilon_n^H\sqrt{-\log\varepsilon_n}$, and we have by Lemma 4.7 (using here $H < K$),

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sup_{\substack{t,s\in[0,1]\\ |t-s|<\varepsilon_n/\delta_n}}\left|X^{(N_n)}(t) - X^{(N_n)}(s)\right| \le \frac{1}{L}\left(\frac{\varepsilon_n}{\delta_n}\right)^H\sqrt{-\log\varepsilon_n}\right) < \infty.$$

Note that $X^{(N_n)}$ is H-self-similar, so that

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sup_{\substack{t,s\in[0,\delta_n]\\ |t-s|<\varepsilon_n}}\left|X^{(N_n)}(t) - X^{(N_n)}(s)\right| \le \frac{1}{L}\phi(n)\right) < \infty. \tag{48}$$

Due to the continuity of W_H, there is an a.s. finite random variable $K = K(\omega)$ such that

$$\max\left\{W_H(1), \sup_{t\in[0,1]}B_H(t)\right\} < K.$$

Consider a number $L_{13} > 0$ and the event $E_1 := \{\max\{W_H(1), \sup_{t\in[0,1]}B_H(t)\} < L_{13}\}$, which has positive probability for all L_{13}. Thus on this event, by the triangle inequality, for $t < \delta_n$,

$$\begin{aligned}|Y(t) - X^{(N_n)}(t)| &\le \sum_{k=N_n}^{\infty}\alpha^k B_H(\{b^k t\}) + \sum_{k=0}^{N_n-1}\alpha^k|W_H(b^k t) - B_H(b^k t)|\\ &\le LL_{13}\alpha^{N_n} + L_{13}\sum_{k=0}^{N_n-1}\alpha^k(b^k\delta_n)^\tau = o(\phi(n)).\end{aligned} \tag{49}$$

It follows from (48), (49), the triangle inequality, and the Borel–Cantelli lemma that on the event E_1, eventually almost surely

$$\sup_{\substack{t,s\in[0,\delta_n]\\ |t-s|<\varepsilon_n}}|Y(t) - Y(s)| > \frac{1}{L}\phi(n).$$

We then have

$$\mathbb{P}\left(\lim_{n\to\infty}\sup_{\substack{t,s\in[0,\delta_n]\\ |t-s|<\varepsilon_n}}|Y(t) - Y(s)| > \frac{1}{L}\phi(n)\right) \ge \mathbb{P}(E_1) > 0.$$

Since ε_n forms a geometric sequence, we can extend the limit to a continuous one. That is,

$$\mathbb{P}\Bigg(\lim_{h\to 0}\sup_{\substack{t,s\in[0,1]\\|t-s|<h}}\frac{|Y(t)-Y(s)|}{h^H\sqrt{-\log h}}>\frac{1}{L}\Bigg)\geq \mathbb{P}(E_1). \tag{50}$$

Recall from Lemma 3.7 that $|t-s|^H/L\leq \big\|Y(t)-Y(s)\big\|_2\leq L|t-s|^H$. Theorem 7.2.1 of [29] then implies that

$$\mathbb{P}\Bigg(\lim_{n\to\infty}\sup_{\substack{t,s\in[0,1]\\|t-s|<h}}\frac{Y(t)-Y(s)}{h^H\sqrt{-\log h}}\leq L\Bigg)=1. \tag{51}$$

Lemma 4.6 together with (50) and (51) then complete the proof.

(ii) Consider the scale $\varepsilon_n=\alpha^n=b^{-nH}$ and $N_n=nH$. We consider three large numbers L_{14},L_{15},L_{16} to be determined later,[6] and for each n consider a number j_n and for $1\leq j\leq j_n$ define

$$s_{n,j}:=b^{-n(H-L_{14}^{-1})-1}+\frac{jL_{16}\varepsilon_n}{L_{15}}\quad\text{ and }\quad t_{n,j}:=b^{-n(H-L_{14}^{-1})-1}+\frac{(jL_{16}+1)\varepsilon_n}{L_{15}}=s_{n,j}+\frac{\varepsilon_n}{L_{15}}.$$

We pick the largest integer j_n such that $t_{n,j_n}<b^{-n(H-L_{14}^{-1})}$. Observe that

$$j_n\geq\frac{1}{L}b^{-n(H-L_{14}^{-1})+nH}=\frac{1}{L}b^{L_{14}^{-1}n}.$$

For intuition, the reader may compare this with the proof of Lemma 4.7, the intervals $[s_j,t_j]$ are now "well-spaced" subintervals of $[b^{-n(H-L_{14}^{-1})-1},b^{-n(H-L_{14}^{-1})}]$.

Consider the truncated Wiener–Weierstrass process

$$Y^{(n)}(t):=\sum_{m=0}^{n-1}\alpha^m B_H(\{b^m t\}).$$

Define for $1\leq j\leq j_n$,

$$\Delta^Y_{n,j}:=Y^{(N_n)}(t_{n,j})-Y^{(N_n)}(s_{n,j})\quad\text{ and }\quad\Delta^B_{n,m,j}:=B_H(\{b^m t_{n,j}\})-B_H(\{b^m s_{n,j}\}). \tag{52}$$

Let $M_n:=n(H-L_{14}^{-1})$, so that for $0\leq m\leq M_n$, $\Delta^B_{n,m,j}=B_H(b^m t_{n,j})-B_H(b^m s_{n,j})$.

By definition and Minkowski's inequality, for $1\leq k<\ell\leq j_n$,

$$\begin{aligned}\Big\|\Delta^Y_{n,k}-\Delta^Y_{n,\ell}\Big\|_2&=\Big\|\sum_{m=0}^{N_n-1}\alpha^m(\Delta^B_{n,m,k}-\Delta^B_{n,m,\ell})\Big\|_2\\&\geq\Big\|\sum_{m=0}^{M_n-1}\alpha^m(\Delta^B_{n,m,k}-\Delta^B_{n,m,\ell})\Big\|_2-\Big\|\sum_{m=M_n}^{N_n-1}\alpha^m\Delta^B_{n,m,k}\Big\|_2-\Big\|\sum_{m=M_n}^{N_n-1}\alpha^m\Delta^B_{n,m,\ell}\Big\|_2.\end{aligned} \tag{53}$$

Our first goal is to prove the lower bound

$$\mathbb{E}\big[|\Delta^Y_{n,k}-\Delta^Y_{n,\ell}|^2\big]\geq\frac{1}{L}\varepsilon_n^{2H}(-\log\varepsilon_n),$$

[6]We will later see that L_{15} depends on L_{16} and L_{14} depends on L_{15},L_{16}.

where L may depend on L_{14}, L_{15}, L_{16}. We first bound $\|\sum_{m=M_n}^{N_n-1} \alpha^m \Delta^B_{n,m,k}\|_2$ from above. Consider the sets

$$T_n := \{m \in [M_n, N_n - 1] \cap \mathbb{Z} : \ \{b^m s_{n,j}\} \le \{b^m t_{n,j}\}\} \quad \text{and} \quad S_n := [M_n, N_n - 1] \cap \mathbb{Z} \setminus T_n.$$

Using the bridge relation (3) and Minkowski's inequality, we write

$$\begin{aligned}
&\Big\| \sum_{m=M_n}^{N_n-1} \alpha^m \Delta^B_{n,m,j} \Big\|_2 \\
&= \Big\| \sum_{m=M_n}^{N_n-1} \alpha^m (W_H(\{b^m t_{n,j}\}) - W_H(\{b^m s_{n,j}\}) - (\kappa(\{b^m t_{n,j}\}) - \kappa(\{b^m s_{n,j}\}))W_H(1)) \Big\|_2 \\
&\le \Big\| \sum_{m\in T_n} \alpha^m (W_H(\{b^m t_{n,j}\}) - W_H(\{b^m s_{n,j}\})) \Big\|_2 + \Big\| \sum_{m\in T_n} \alpha^m (\kappa(\{b^m t_{n,j}\}) - \kappa(\{b^m s_{n,j}\}))W_H(1) \Big\|_2 \\
&\qquad + \Big\| \sum_{m\in S_n} \frac{\alpha^m b^m \varepsilon_n}{L_{15}} (W_H(\{b^m t_{n,j}\}) + W_H(1) - W_H(\{b^m s_{n,j}\})) \Big\|_2 \\
&\qquad + \Big\| \sum_{m\in S_n} \alpha^m (\kappa(\{b^m t_{n,j}\}) + 1 - \kappa(\{b^m s_{n,j}\}))W_H(1) \Big\|_2 .
\end{aligned}$$

The second term is bounded by

$$\Big\| \sum_{m=M_n}^{N_n-1} \alpha^m (\kappa(\{b^m t_{n,j}\}) - \kappa(\{b^m s_{n,j}\}))W_H(1) \Big\|_2 \le \sum_{m=M_n}^{N_n-1} \alpha^m \left(\frac{b^m \varepsilon_n}{L_{15}} \right)^\tau \le L\varepsilon_n^H.$$

To estimate the first term, we define

$$f_n(t) := \sum_{m\in T_n} \alpha^m \mathbb{1}_{[\{b^m s_{n,j}\}, \{b^m t_{n,j}\}]}(t),$$

so that

$$\Big\| \sum_{m=M_n}^{N_n-1} \alpha^m (W_H(\{b^m t_{n,j}\}) - W_H(\{b^m s_{n,j}\})) \Big\|_2^2 = \mathbb{E}\left[\left| \int_0^\infty f_n(t)\, dW_H(t) \right|^2 \right].$$

By Theorem 3.3 and self-similarity of W_H,

$$\mathbb{E}\left[\left| \int_0^\infty f_n(t)\, dW_H(t) \right|^2 \right] \le L(N_n - M_n)\varepsilon_n^{2H} \le \frac{Ln\varepsilon_n^{2H}}{L_{14}}$$

where L does not depend on L_{14}. Similar estimates hold for the third and fourth terms using the case $k = 2$ of Theorem 3.3. We then conclude that

$$\Big\| \Delta^Y_{n,k} - \Delta^Y_{n,\ell} \Big\|_2 \ge \Big\| \sum_{m=0}^{M_n-1} \alpha^m (\Delta^B_{n,m,k} - \Delta^B_{n,m,\ell}) \Big\|_2 - \frac{L\sqrt{n}\varepsilon_n^H}{L_{14}}. \tag{54}$$

Next, we bound $\|\sum_{m=0}^{M_n-1} \alpha^m (\Delta^B_{n,m,k} - \Delta^B_{n,m,\ell})\|_2$ from below. We first define $\Delta^W_{n,m,k}$ similarly as in (52), so by a similar Minkowski's inequality argument, it suffices to give a lower bound for

$$\Big\| \sum_{m=0}^{M_n-1} \alpha^m (\Delta^W_{n,m,k} - \Delta^W_{n,m,\ell}) \Big\|_2 .$$

This is done again by an induction argument. Consider first the term with $m = 0$. We compute

$$\begin{aligned}\mathbb{E}[|\Delta^W_{n,0,k} - \Delta^W_{n,0,\ell}|^2] &= \mathbb{E}[|\Delta^W_{n,0,k}|^2] + \mathbb{E}[|\Delta^W_{n,0,\ell}|^2] - 2\mathbb{E}[\Delta^W_{n,0,k}\Delta^W_{n,0,\ell}] \\ &\geq 2\left(\frac{\varepsilon_n}{L_{15}}\right)^{2H} - 2L\left(\frac{\varepsilon_n}{L_{15}}\right)^2\left(\frac{L_{16}\varepsilon_n}{L_{15}}\right)^{2H-2} = \left(\frac{2}{L_{15}^{2H}} - \frac{2LL_{16}^{2H-2}}{L_{15}^{2H}}\right)\varepsilon_n^{2H},\end{aligned}$$

where the last line follows from the mean-value theorem and the number L does not depend on L_{15}, L_{16}. Picking L_{16} large enough we see that

$$\mathbb{E}\big[|\Delta^W_{n,0,k} - \Delta^W_{n,0,\ell}|^2\big] \geq \frac{\varepsilon_n^{2H}}{LL_{15}^{2H}}. \tag{55}$$

Consider the case $1 \leq M \leq M_n - 1$. By self-similarity,

$$\mathbb{E}\big[|\alpha^M(\Delta^W_{n,M,k} - \Delta^W_{n,M,\ell})|^2\big] \geq \frac{\varepsilon_n^{2H}}{LL_{15}^{2H}}.$$

Thus we have

$$\begin{aligned}&\mathbb{E}\Big[\Big|\sum_{m=0}^{M}\alpha^m(\Delta^W_{n,m,k} - \Delta^W_{n,m,\ell})\Big|^2\Big] - \mathbb{E}\Big[\Big|\sum_{m=0}^{M-1}\alpha^m(\Delta^W_{n,m,k} - \Delta^W_{n,m,\ell})\Big|^2\Big] \\ &= \mathbb{E}\left[\Big|\alpha^M(\Delta^W_{n,M,k} - \Delta^W_{n,M,\ell})\Big|^2\right] + 2\mathbb{E}\Big[\alpha^M(\Delta^W_{n,M,k} - \Delta^W_{n,M,\ell})\Big(\sum_{m=0}^{M-1}\alpha^m(\Delta^W_{n,m,k} - \Delta^W_{n,m,\ell})\Big)\Big] \\ &\geq \frac{\varepsilon_n^{2H}}{LL_{15}^{2H}} + 2\alpha^M\mathbb{E}\Big[(\Delta^W_{n,M,k} - \Delta^W_{n,M,\ell})\Big(\sum_{m=0}^{M-1}\alpha^m(\Delta^W_{n,m,k} - \Delta^W_{n,m,\ell})\Big)\Big].\end{aligned} \tag{56}$$

Consider first $0 < H \leq 1/2$ where the increments of W_H are negatively correlated, then

$$\begin{aligned}&\mathbb{E}\Big[(\Delta^W_{n,M,k} - \Delta^W_{n,M,\ell})\Big(\sum_{m=0}^{M-1}\alpha^m(\Delta^W_{n,m,k} - \Delta^W_{n,m,\ell})\Big)\Big] \\ &\geq \mathbb{E}\Big[\Delta^W_{n,M,k}\Big(\sum_{m=0}^{M-1}\alpha^m\Delta^W_{n,m,k}\Big)\Big] + \mathbb{E}\Big[\Delta^W_{n,M,\ell}\Big(\sum_{m=0}^{M-1}\alpha^m\Delta^W_{n,m,\ell}\Big)\Big] \\ &= \sum_{m=0}^{M-1}\alpha^m\mathbb{E}\left[\Delta^W_{n,M,k}\Delta^W_{n,m,k} + \Delta^W_{n,M,\ell}\Delta^W_{n,m,\ell}\right].\end{aligned} \tag{57}$$

Observe that for $m < M$ and $1 \leq j \leq j_n$, $b^m t_{n,j} < b^M s_{n,j} - b^M/L$, so that by mean-value theorem (recalling $t_{n,j} - s_{n,j} = \varepsilon_n/L_{15}$)

$$|\mathbb{E}[\Delta^W_{n,M,k}\Delta^W_{n,m,k}]| \leq L\Big(\frac{b^M}{L}\Big)^{2H-2} b^{m+M}\frac{\varepsilon_n^2}{L_{15}^2}, \tag{58}$$

where L does not depend on L_{15}. Using the relation $M < nH$ and $\varepsilon_n = \alpha^n = b^{-nH}$, and combining (56), (57), and (58), we have for L_{15} large enough,

$$\mathbb{E}\Big[\Big|\sum_{m=0}^{M}\alpha^m(\Delta^W_{n,m,k} - \Delta^W_{n,m,\ell})\Big|^2\Big] - \mathbb{E}\Big[\Big|\sum_{m=0}^{M-1}\alpha^m(\Delta^W_{n,m,k} - \Delta^W_{n,m,\ell})\Big|^2\Big] \geq \frac{\varepsilon_n^{2H}}{L}.$$

By summing over the previous relation and using (55), we obtain

$$\mathbb{E}\Big[\Big|\sum_{m=0}^{M}\alpha^m(\Delta^W_{n,m,k}-\Delta^W_{n,m,\ell})\Big|^2\Big]\geq\frac{M\varepsilon_n^{2H}}{L}\geq\frac{1}{L}\varepsilon_n^{2H}(-\log\varepsilon_n).$$

A similar argument works for $H>1/2$. Combining with (54) and (53), we obtain that for L_{14} large enough,

$$\Big\|\Delta^Y_{n,k}-\Delta^Y_{n,\ell}\Big\|_2\geq\frac{1}{L}\varepsilon_n^H\sqrt{-\log\varepsilon_n}.$$

Therefore, by the Sudakov minoration,

$$\mathbb{E}\Big[\sup_{1\leq j\leq j_n}\Delta^Y_{n,j}\Big]\geq\frac{1}{L}\left(\frac{1}{L}\varepsilon_n^H\sqrt{-\varepsilon_n}\right)\sqrt{\log j_n}\geq\frac{1}{L}\varepsilon_n^H(-\log\varepsilon_n).$$

Our next goal is to bound $\|\Delta^Y_{n,j}\|_2$ from above. Using Minkowski's inequality and Lemma 3.8,

$$\Big\|\Delta^Y_{n,j}\Big\|_2\leq\big\|Y(t_{n,j})-Y(s_{n,j})\big\|_2+\Big\|\widetilde{Y}^{(N_n)}(t_{n,j})-\widetilde{Y}^{(N_n)}(s_{n,j})\Big\|_2\leq Ln\varepsilon_n^H+\Big\|\widetilde{Y}^{(N_n)}(t_{n,j})-\widetilde{Y}^{(N_n)}(s_{n,j})\Big\|_2,$$

where

$$\widetilde{Y}^{(N_n)}(t)=\sum_{m=N_n}^{\infty}\alpha^m B_H(\{b^m t\}).$$

By a similar argument as in (49) and Gaussian concentration,

$$\Big\|\widetilde{Y}^{(N_n)}(t_{n,j})-\widetilde{Y}^{(N_n)}(s_{n,j})\Big\|_2\leq L\varepsilon_n^H,$$

and hence, $\|\Delta^Y_{n,j}\|_2\leq Ln\varepsilon_n^H$. By the Gaussian concentration inequality,

$$\mathbb{P}\left(\Big|\sup_{1\leq j\leq j_n}\Delta^Y_{n,j}-\mathbb{E}\Big[\sup_{1\leq j\leq j_n}\Delta^Y_{n,j}\Big]\Big|\geq u\right)\leq L\exp\left(\frac{-u^2}{Ln\varepsilon_n^{2H}}\right).$$

Choose $H'\in(\max\{2H,1\},2)$. Taking $u=\varepsilon_n^H(-\log\varepsilon_n)^{H'}$ and using the trivial bound

$$\sup_{\substack{t,s\in[0,1]\\|t-s|<\varepsilon_n}}\Big(Y^{(N_n)}(t)-Y^{(N_n)}(s)\Big)\geq\sup_{1\leq j\leq j_n}\Delta^Y_{n,j}$$

yield that

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sup_{\substack{t,s\in[0,1]\\|t-s|<\varepsilon_n}}\Big(Y^{(N_n)}(t)-Y^{(N_n)}(s)\Big)\leq\frac{1}{L}\varepsilon_n^H(-\log\varepsilon_n)\right)<\infty.$$

Consider a large number L_{17} and the non-trivial event

$$E_2:=\left\{\sup_{0\leq t\leq 1}B_H(t)\leq L_{17}\right\}.$$

Thus on this event,

$$\Big|\Big(Y^{(N_n)}(t)-Y^{(N_n)}(s)\Big)-\big(Y(t)-Y(s)\big)\Big|\leq LL_{17}\varepsilon_n^H.$$

By the Borel–Cantelli lemma, on E_2 we have eventually a.s.

$$\sup_{\substack{t,s\in[0,1]\\|t-s|<\varepsilon_n}} |Y(t)-Y(s)| > \frac{1}{L}\varepsilon_n^H(-\log\varepsilon_n).$$

Since $\varepsilon_n = \alpha^n$ forms a geometric sequence, we obtain

$$\mathbb{P}\left(\lim_{n\to\infty}\sup_{\substack{t,s\in[0,1]\\|t-s|<h}} \frac{Y(t)-Y(s)}{h^H(-\log h)} > \frac{1}{L}\right) \geq \mathbb{P}(E_2) > 0. \tag{59}$$

Recall from Lemma 3.8 that $|t-s|^H\sqrt{-\log|t-s|}/L \leq \left\|Y(t)-Y(s)\right\|_2 \leq L|t-s|^H\sqrt{-\log|t-s|}$. Theorem 7.2.1 of [29] then implies that

$$\mathbb{P}\left(\lim_{n\to\infty}\sup_{\substack{t,s\in[0,1]\\|t-s|<h}} \frac{Y(t)-Y(s)}{h^H(-\log h)} \leq L\right) = 1. \tag{60}$$

Lemma 4.6 together with (59) and (60) then complete the proof.

(iii) This follows directly from Theorem 2.3(iii). □

Proof of Theorem 2.6. The upper bound of the Hausdorff dimension follows from pathwise Hölder continuity (Proposition 2.3 of [10] and Proposition A.1 of [40]). To show the lower bound, first note that by Lemma 3.7, if $H<K$, there exists $L\in\mathbb{N}$ such that

$$\mathbb{E}\left[\left|\frac{Y(t)-Y(s)}{|t-s|^H}\right|^2\right] \geq \frac{1}{L} \quad \text{for} \quad (s,t)\in[0,1]^2,\ |s-t|<b^{-L}.$$

Thus by Gaussianity, there exists some $a>0$ such that for all $0\leq k<b^L$,

$$\sup_{s,t\in[kb^{-L},(k+1)b^{-L}]} \mathbb{P}\left(-x\leq \frac{Y(t)-Y(s)}{|t-s|^H}\leq x\right) \leq ax \quad \text{as} \quad x\to 0^+.$$

By Theorem 2 of [26], the Hausdorff dimension of the graph Y on $[kb^{-L},(k+1)b^{-L}]$ is bounded from below by $2-H$ almost surely. By countable stability of the Hausdorff dimension (see Section 2.2 of [10]), we must have $\dim(Y)\geq 2-H$ on $[0,1]$. The case $H=K$ is similar using Lemma 3.8 instead of Lemma 3.7.

The case $H>K$ requires more work. Recall from (6) that $K=\min\{1,(-\log_b\alpha)\}=-\log_b\alpha$, so it suffices to prove for a fixed $\varepsilon>0$ that $\dim(Y)\geq 2+\log_b(\alpha)-2\varepsilon$. Let us fix $b\in\mathbb{N}$, $H\in(0,1)$, $\alpha\in(0,1)$ with $\alpha b^{2H-1}<1$, and $\varepsilon>0$. By the potential-theoretic lower bound (see Section 4.3 of [10]), it suffices to find a probability measure $\nu=\nu_\omega$ on the graph G_ω of Y such that

$$I_\omega := \int_{G_\omega}\int_{G_\omega} \frac{d\nu(t)\,d\nu(s)}{|t-s|^\gamma} < \infty \quad \text{a.s.} \tag{61}$$

Recall (35). Let N be a large even number such that $b^{N(1-\varepsilon)}<b^N-2$. Define a set $S_N\subseteq[0,1]$ by its $b^{N/2}$-adic expansion, such that $x=\sum_{i=1}^\infty \xi_i b^{-iN/2}\in S_N$ if $\xi_i\in\{1,2,\ldots,b^{N/2}-2\}$ for all i. It is elementary to check that $S_N\subseteq T_N$. Moreover, the set S_N in Lemma 3.9 is a Cantor-like self-similar set. We may define the uniform measure on S_N, similarly to the construction of the Cantor measure

(see Example 17.1 of [10]). This can be done, for instance, by choosing each decimal ξ_i independently and uniformly from $\{1, \dots, b^{N/2} - 2\}$. Denote such a measure by μ_N. It is then standard to verify that if $b^{N(1-\varepsilon)/2} < b^{N/2} - 2$, then

$$\int_{T_N} \int_{T_N} |t-s|^{\varepsilon - 1} d\mu_N(t)\, d\mu_N(s) < \infty; \tag{62}$$

see, for instance, Exercise 4.11 of [10] for the Cantor set.

We let ν be the pushforward of μ_N under the map $t \mapsto (t, X(t, \omega))$, thus reducing (61) to proving that

$$\mathbb{E}[I_\omega] = \int_{T_N} \int_{T_N} \mathbb{E}\left[\left(|t-s|^2 + |Y(t) - Y(s)|^2 \right)^{-\gamma/2} \right] d\mu_N(t)\, d\mu_N(s) < \infty.$$

By Lemma 3.9 and a similar computation as in the proof of Theorem 2(i) of [26], we obtain

$$\mathbb{E}\left[\left(|t-s|^2 + |Y(t) - Y(s)|^2 \right)^{-\gamma/2} \right] \le L|t-s|^{-\gamma + 1 + \log_b(\alpha)} \le L|t-s|^{\varepsilon - 1}.$$

Combining with (62) yields $\mathbb{E}[I_\omega] < \infty$, and hence the proof is complete. □

4.3 Distribution of the maximum location

Proof of Theorem 2.7. Suppose first $H \le K$, we prove that for all $s \in [0,1]$, $\mathbb{P}(s \text{ is a local maximum}) = 0$. By arguing similarly as in the proof of (b) and (c) of Theorem 2.3 and applying Lemma 4.6, for $H = K$, there exists a deterministic function $C_1 : [0,1] \to \mathbb{R}_+$ such that for all $s \in [0,1]$,

$$\mathbb{P}\left(\liminf_{t \in [0,1],\, t \to s} \frac{Y(t) - Y(s)}{\sqrt{2|t-s|^{2H} \log(1/|t-s|) \log\log(1/|t-s|)}} = -C_1(s) \right) = 1,$$

and for $H < K$, there exists $C_2 : [0,1] \to \mathbb{R}_+$ such that

$$\mathbb{P}\left(\liminf_{t \in [0,1],\, t \to s} \frac{Y(t) - Y(s)}{\sqrt{2|t-s|^{2H} \log\log(1/|t-s|)}} = -C_2(s) \right) = 1.$$

This along with parts (b) and (c) of Theorem 2.3 proves $\mathbb{P}(s \text{ is a local maximum}) = 0$.

For the remainder of the proof, let $H > K$. We prove that $\mathbb{P}(0 \text{ is a global maximum}) > 0$. We make an auxiliary claim that there is a Lipschitz function $\phi : [0,1] \to \mathbb{R}$ such that[7]

- $\phi(0) = \phi(1) = 0, \phi(t) > 0$ for all $t \in (0,1)$. In particular, ϕ attains its minimum at $0, 1$;
- $\phi \in \mathcal{H}_H$ where $\mathcal{H}_H$ is the Cameron–Martin space of the fractional Brownian motion with parameter H.

Provided the claim is true, we consider a large number L and let

$$f(t) := L \sum_{m=0}^{\infty} \alpha^m \phi(\{b^m t\})$$

be the fractal function associated with $L\phi$. It is easy to see that $f(t)$ attains its minima at $0, 1$, and[8]

$$f(t) \ge L(t \wedge (1-t))^K. \tag{63}$$

[7] $\phi(t) = 1 - \cos(2\pi t)$ may work, but we take the shortest path.

[8] Of course, the L here may not be the same and may depend on the choice of ϕ.

Consider the process

$$\widetilde{Y}(t) := Y(t) + f(t) = \sum_{m=0}^{\infty} \alpha^m \widetilde{B}_H(\{b^m t\}),$$

so that $Y(t) = \widetilde{Y}(t) - f(t)$. It suffices to show $\widetilde{Y}(t)$ is Hölder continuous with exponent $-\log_b(\alpha)$ and constant $< 1/L$ with positive probability, because this implies $\mathbb{P}(|\widetilde{Y}(t)| \leq \frac{1}{L}(t \wedge (1-t))^K) > 0$, which along with (63) yields $\mathbb{P}(Y(t) \leq 0, \forall t \in [0,1]) > 0$.

We recall in the proof of Proposition A.1 of [40] and the uniform modulus of continuity of fractional Brownian bridge that there exists $\delta_1 > 0$ such that we have the inclusion of events

$$\left\{ \sup_{t \in [0,1]} |\widetilde{B}_H(t)| \leq \delta_1 \right\} \subseteq \left\{ \widetilde{Y}(t) \text{ is } -\log_b(\alpha)\text{-Hölder with constant } \frac{1}{L} \right\}.$$

By the bridge relation and since $\widetilde{B}_H(t) = B_H(t) + L\phi(t)$, there is $\delta_2 > 0$ depending on κ such that

$$\left\{ \sup_{t \in [0,1]} |W_H(t) + L\phi(t)| \leq \delta_2 \right\} \subseteq \left\{ \sup_{t \in [0,1]} |B_H(t) + L\phi(t)| \leq \delta_1 \right\} \subseteq \left\{ \sup_{t \in [0,1]} |\widetilde{B}_H(t)| \leq \delta_1 \right\}.$$

Since $\phi \in \mathcal{H}_H$, so does $-L\phi$. This implies the probability of the event on the left-hand side is positive by Theorem 2 of [28] (or by the Cameron–Martin Theorem for the fractional Brownian motion). Combining the above gives

$$\mathbb{P}\big(\widetilde{Y}(t) \text{ is } -\log_b(\alpha)\text{-Hölder continuous with constant } 1/L\big) > 0.$$

We now prove the auxiliary claim. The function ϕ will arise from ϕ_1 of the form

$$\phi_1(t) := \int_0^t (t-s)^{H-1/2} \psi_1(s)\, ds$$

where $\psi_1(s) = s(1-s)$. It is easy to see that there is $T > 0$ with $\phi_1(0) = \phi_1(T) = 0$ and ϕ_1 is positive and Lipschitz on $[0,T]$. Define $\psi(s) := \psi_1(Ts)$ and

$$\phi(t) := \int_0^t (t-s)^{H-1/2} \psi(s)\, ds,$$

by a change of variable we obtain the first item of the claim; that $\phi \in \mathcal{H}_H$ follows from Theorem 5.4 of [35] and the fact that ψ is square-integrable. ☐

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